Lecture Notes from September 13, 2022

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Warm up:

1.47 Question. : Given an isometry $V : H \to H$, show that VV^* is an orthogonal projection.

Here, the given map VV^* satisfies $< VV^*x, y > = < V^*x, V^*y >$

Consider

$$\begin{aligned}(VV^*)^* &= (V^*)^*V^* \\ &= VV^*\end{aligned}$$

Thus, VV^* is Hermitian. (by definition)
We also know by assumption, $VV^* = id_H$
We can then see that

$$\begin{aligned}VV^*VV^* &= V(V^*V)V^* \\ &= V(id_H)V^* \\ &= VV^*\end{aligned}$$

Therefore, VV^* is an orthogonal projection on a Hilbert space

We leave showing that this is an orthogonal projection to a result in this class.

1.48 Theorem. *For a bounded linear map $A : H_1 \to H_2$, the following are equivalent:*

a) A is unitary

b) A is onto and preserves the inner product, i.e, for $x, y \in H$, $< Ax, Ay > = < x, y >$

c) A is the bijection and preserves the inner product

d) A is onto and an isometry

Proof. $a) \to b)$
Since A is unitary, we know that A has an isometry and surjection. Thus, for each $x \in H_2$, we know that $AA^* = id_H$ and thus, $AA^*x = x$.

So, we can re-write it as $A(A^*)x = x$.
Hence, A is surjective (onto).
Now, the inner product is invariant by

$$\begin{aligned} < Ax, Ay > &= < A^*Ax, y > \\ &= < x, y > \end{aligned}$$

Thus, A is onto and preserves the inner product.

$b) \to c)$
Here, given that A is onto and preserves the inner product. Now, A is 1-1 follows from

$$\begin{aligned} (||Ax||)^2 &= < Ax, Ax > \\ &= < x, x > \\ &= (||x||)^2 \end{aligned}$$

So, $\quad Ax = 0$
This implies that $x = 0$.
Hence, A is one-one (injective)
Thus, A is the bijection and preserves the inner product.

$c) \to d)$
Since A is a bijection, we know that A is one-one and A is onto.
From the preservation of the inner product by $(||Ax||)^2 = (||x||)^2$ as in the proof of (c), we can see that the isometry property follows
(Here, $(||Ax||)^2 = (||x||)^2$ is the direct consequence of inner product being preserved.)
Thus, A is onto and also an isometry.

$d) \to a)$
From the isometry assumption, we know that, since A is an isometry, $A^*A = id_{H1}$ (from Lemma 1.49)
Then, we have,

$$\begin{aligned} A^*A &= id_{H1} \\ AA^*A &= A \end{aligned}$$

Since A is onto, $AA^* = id_{H2}$
Therefore, A is unitary.

$\square$

We could if needed extend, these equivalences to bijections between inner product spaces. Next, we see the characterization of normality in geometric terms, with the norm of image vectors.

1.49 Lemma. *An operator $A \in B(H)$ is normal iff for each $x \in H$, $||Ax|| = ||A^*x||$*

Proof. Assume $||Ax|| = ||A^*x||$

Now, we have,

$$
\begin{aligned}
< (AA^* - A^*A)x, x > &=< (AA^*x, x > - < A^*A)x, x > \text{(Since Hermitian)} \\
&=< (A^*x, A^*x > - < Ax, Ax > \\
&= (||A^*x||)^2 - (||Ax||)^2 \\
&= 0
\end{aligned}
$$

Now, we know that AA^* and A^*A are Hermitian.
Thus, we have, AA^* - $A^*A = 0$ because its quadratic form vanishes.

Conversely, Suppose that $AA^* = A^*A$.
Here, we see this is true, which implies $(||A^*x||)^2 = (||Ax||)^2$
Thus, $||A^*x|| = ||Ax||$
Therefore, $A \in B(H)$ is normal iff for each $x \in H$, $||Ax|| = ||A^*x||$

□

Next, we study about how the adjoint of operator, range and kernel relates.

We write $\mathcal{N}(A)$ for the null space $\mathcal{N}(A) = A^{-1}(\{0\})$
and $\mathcal{R}(A)$ for the range $\mathcal{R}(A) = A(\mathcal{H})$

1.50 Lemma. *For $A \in \mathcal{B}$ $(\mathcal{H})$,*

1. *$\mathcal{N}(A) = \mathcal{R}(A^*)^{\perp}$*
2. *A closed subspace E is invariant under A, i.e, $A(E) \subset E$ if and only if $E^{\perp}$ is invariant under A^**

Proof. :
1) We know that $< Ax, y > = < x, A^*y >$,
Thus, Ax = 0. This is equivalent to $x \in (R(A^*))^{\perp}$

2) Assume that $A(E) \subset E$. Then for $v \in E^{\perp}$, $y \in E$, we have,

$$
\begin{aligned}
< A^*v, y > &=< v, Ay > \\
&= 0
\end{aligned}
$$

Since Ay $\in$ E and E is invariant.
Thus, $A^*(E^\perp) \subset E^\perp$ (because if we apply something to E, it's in E$\perp$)

Now, Conversely, Suppose $A^*(E^\perp) \subset E^\perp$.
Then since E is a closed subspace,
E = $(E^\perp)^\perp$
and
A = $(A^*)^*$
Thus, switching A and A^* in the preceeding result, we derive $A(E) \subset E$.

□

Finally, we will characterize the orthogonal projections

1.51 Theorem. *Let $0 \neq \mathcal{P} \in \mathcal{B}(\mathcal{H})$ be a projection, i.e. $\mathcal{P}^2 = \mathcal{P}$, then the following are equivalent:*

1. *$\mathcal{P}$ is an orthogonal projection, so $\mathcal{N}(\mathcal{P}) \perp \mathcal{R}(\mathcal{P})$*
2. *$||\mathcal{P}|| = 1$*
3. *$< \mathcal{P}x, x > \geq 0$ for each $x \in \mathcal{H}$*
4. *$\mathcal{P} = \mathcal{P}^*$*
5. *$\mathcal{P}$ is normal*

Proof. :
We recall that if $\mathcal{P}^* = \mathcal{P}$. Then, we know that $\mathcal{H} = \mathcal{R}(\mathcal{P}) \bigoplus \mathcal{N}(\mathcal{P})$, because any vector x can be expressed as x = $\mathcal{P}x$ + $(\mathcal{I} - \mathcal{P})$x.
Since $\mathcal{P}$ is bounded, we see that both subspaces are closed and for any projection operator $\mathcal{P}$, we know that $(\mathcal{I} - \mathcal{P})^2 = (\mathcal{I} - \mathcal{P})$.
So, Im($\mathcal{P}$) = Ker $(\mathcal{I} - \mathcal{P})$.
Now, if we apply this to our above projection operator, we will get, ker($\mathcal{P}$) = Im $(\mathcal{I} - \mathcal{P})$
Thus, when we express any vector x as x = $\mathcal{P}x$ + $(\mathcal{I} - \mathcal{P})$x, we have, $\mathcal{P}$x is the image of $\mathcal{P}$ and $(\mathcal{I} - \mathcal{P})$x is in Ker($\mathcal{P}$).
Here, $\mathcal{N}(\mathcal{P})$ is closed because it is $\mathcal{P}^{-1}$ of other vector. i.e, $\mathcal{N}(\mathcal{P}) = \mathcal{P}^{-1}$ ($\{0\}$)
and, $\mathcal{R}(\mathcal{P})$ is identity of other vector i.e, $\mathcal{R}(\mathcal{P}) = (\mathcal{I} - \mathcal{P})^{-1}(\{0\})$
So, we have $\mathcal{P}x \perp (I - \mathcal{P})x$

Now, 1) $\rightarrow$ 2)

Let E = $\mathcal{R}(\mathcal{P})$
By assumption, we have, $\mathcal{H}$ = E $\bigoplus$ $E^\perp$ with N(P) = $E^\perp$
From $\mathcal{P} \neq 0$, E $\neq \{0\}$

Thus, there is $x \in E$, $||x|| = 1$

$$
\begin{aligned}
\text{By,} \quad & \mathcal{P}^2 = \mathcal{P}, \\
& \mathcal{P}x = x, \qquad x \in \mathcal{R}(\mathcal{P}) \\
\text{or,} \quad & ||\mathcal{P}x|| = ||x|| = 1 \\
\text{So,} \quad & ||\mathcal{P}|| \geq 1
\end{aligned}
$$

On the other hand, given $x \in \mathcal{H}$, then $x = y + z$ with $y \in E$, $z \in E^{\perp}$ and

$$
\begin{aligned}
(||\mathcal{P}x||)^2 &= (||y||)^2 \\
&= (||x||)^2 - (||z||)^2 \\
&\leq (||x||)^2 \quad \text{(pythagorean Identity)} \\
\text{So,} \quad ||P|| &\leq 1
\end{aligned}
$$

Therefore, since $||P|| \leq 1$ and $||P|| \geq 1$, we know that $||P|| = 1$

$\square$