MATH 7320 Lecture Notes

Note-taker: Kumari Teena

September 13, 2022

Warm up: Given an isometry $V : \mathcal{H} \longrightarrow \mathcal{H}$, show that VV^* is an orthogonal projection.

Proof: Consider

$$(VV^*)^* = (V^*)^* V^* = VV^* \ .$$

This implies that VV^* is Hermitian. Since V is an isometry, then by previous Lemma, we have

$$V^* V = id_{\mathcal{H}} \ ,$$

we see that,

$$(VV^*)^2 = VV^*VV^* = VV^*$$

$\implies VV^*$ is a projection.

Now

$$\langle VV^* x, y \rangle = \langle V^* x, V * y \rangle = \langle x, VV^* y \rangle, \quad \forall \quad x, y \in \mathcal{H} \ .$$

Thus, VV^* is an orthogonal projection.

Motivation: Characterize unitaries as "onto isometries "

Theorem 1. *For a bounded linear map $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, the following are equivalent:*

1. *A is unitary.*
2. *A is onto and preserves the inner product, i.e. for $x, y \in \mathcal{H}$,*

$$\langle Ax, Ay \rangle = \langle x, y \rangle \ .$$

3. A is a bijection and preserves the inner product.

4. A is onto and an isometry.

Proof: (1) $\implies$ (2) For each $x \in \mathcal{H}_2$, we know that $AA^*x = x$. So $A(A^*x) = x$ and $A^*x \in \mathcal{H}_1$. Hence A is surjective (onto).
The inner product is invariant by

$$\langle Ax, Ay \rangle = \langle A^*Ax, y \rangle = \langle x, y \rangle, \quad \forall \;\; x, y \in \mathcal{H}_1$$

as $A^*A = id_{\mathcal{H}_1}$.
(2) $\implies$ (3) Suppose A is onto and preserves the inner product. Then

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, x \rangle = \|x\|^2 \ .$$

So $Ax = 0$ if and only $x = 0$. Hence A is one-to-one (injective).
So $Ax = 0 \iff x = 0$. Hence, A is one to one. (3) $\implies$ (4) From A being a bijection, it is onto. The isometry property follows from preservation of the inner product by

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, x \rangle = \|x\|^2 \ .$$

. (4) $\implies$ (1) from the isometry assumption $A^*A = id_{\mathcal{H}_1}$, then $AA^*A = A$ and since A is onto, then for each $x \in \mathcal{H}_2$ there exists $y \in \mathcal{H}_1$ such that $Ay = x$. Now,

$$\begin{aligned} (AA^*A)y &= Ay \quad \implies \quad (AA^*)Ay = Ay \\ (AA^*)x &= x, \;\; \forall \;\; x \in \mathcal{H}_2 \end{aligned}$$

So, we have $AA^* = id_{\mathcal{H}_1}$, We could, if needed, extend these equivalences to bijections between inner product spaces. □

Next we characterise **normality**.

Geometric characterization of normality

Lemma 2. *An operator $A \in \mathcal{B}(\mathcal{H})$ is normal if and only if for for each $x \in \mathcal{H}$, $\|Ax\| = \|A^*x\|$.*

Proof: We have, assuming that $\|Ax\| = \|A^*x\|$, then

$$\begin{aligned} \langle (AA^* - A^*A)x, x\rangle &= \langle AA^*x - A^*Ax, x\rangle \\ &= \langle AA^*x, x\rangle - \langle A^*Ax, x\rangle \\ &= \langle A^*x, A^*x\rangle - \langle Ax, Ax\rangle \\ &= \|A^*x\|^2 - \|Ax\|^2 = 0 \end{aligned}$$

By $AA^* - A^*A$ being Hermitian, we can deduce that $AA^* - A^*A = 0$, because it is quadratic form vanishes. Conversely, if $AA^* - A^*A = 0$ then we can see this is true, which implies that $\|Ax\|^2 = \|A^*x\|^2$.

We write $\mathcal{N}(A)$ for the **null space**, $\mathcal{N} = A^{-1}(\{0\})$ and $\mathcal{R}(A)$ for the **range** $\mathcal{R}(A) = A(\mathcal{H})$.

Relationships between adjoints, null space and range

Lemma 3. *For $A \in \mathcal{B}(\mathcal{H})$*

1. $\mathcal{N}(A) = (\mathcal{R}(A^*))^{\perp}$.
2. *A closed subspace E is invariant under A, i.e. $A(E) \subset E$, if and only if $E^{\perp}$ is invariant under A^*.*

Proof: *(1)* By $\langle Ax, y\rangle = \langle x, A^*y\rangle$. Let $Ax = 0$, this implies that

$$\langle Ax, y\rangle = 0 \implies \langle x, A^*y\rangle = 0 \ \ i.e. \ \ x \perp A^*y \ .$$

Thus, $Ax = 0$ is equivalent to $x \in (\mathcal{R}(A^*))^{\perp}$. So $\mathcal{N}(A) = (\mathcal{R}(A^*))^{\perp}$.

(2) Assuming $A(E) \subset E$, then for $v \in E^{\perp},\ y \in E$

$$\langle A^*v, y\rangle = \langle v, Ay\rangle = 0,\ \forall\ y \in E$$

$\implies \forall\ v \in E^{\perp},\ A^*v \in E^{\perp}$ So, $A^*(E^{\perp}) \subset E^{\perp}$.

Conversely, if $A^*(E^{\perp}) \subset E^{\perp}$, then by E being a closed subspace $E = (E^{\perp})^{\perp}$ and $A = (A^*)^*$, so then switching A and A^* in the preceding result gives $A(E) \subset E$. □

Next we want to characterise orthogonal projection.

Characterization of orthogonal projection

Theorem 4. *Let $0 \neq P \in \mathcal{B}(\mathcal{H})$ be a projection i.e. $P^2 = P$, then the following are equivalent:*

1. *P is an orthogonal projection, so $\mathcal{N}(P) \perp \mathcal{R}(P)$.*
2. *$\|P\| = 1$.*
3. *$\langle Px, x\rangle \geq 0 \ \forall \ x \in \mathcal{H}$.*
4. *$P = P^*$.*
5. *P is normal.*

Proof: We recall that $P^2 = P$, then $\mathcal{H} = \mathcal{R}(P) \bigoplus \mathcal{N}(P)$. So for each $x \in \mathcal{H}$, $x = P(x) + (I - P)x$ as $(I - P)x \in \mathcal{N}(P)$. Since P is bounded, both subspaces are closed which is proved as follows: Let x be a limit point of $\mathcal{N}(P)$, then there is a sequence x_n in space $\mathcal{N}(P)$ such that $x_n \longrightarrow x$. Then for each $y \in \mathcal{H}$

$$\langle P(x_n), y\rangle = \langle x_n, P(y)\rangle \longrightarrow \langle x, P(y)\rangle \ ,$$

$\implies \ \langle x, P(y)\rangle = 0$ as $P(x_n) = 0$. That is $\langle P(x), y\rangle = 0 \ \forall \ \ y \in \mathcal{H}$. This implies that $P(x) = 0$ i.e. $x \in \mathcal{N}(P)$. Therefore, $\mathcal{N}(P)$ is closed.
Next we prove that $\mathcal{R}(P)$ is closed. Let $x \in \mathcal{R}(P)$ and $z \in \mathcal{N}(P)$, then there exists $y \in \mathcal{H}$ such that $P(y) = x$. Now consider

$$\langle x, z\rangle = \langle P(y), z\rangle = \langle y, P(z)\rangle = 0 \ ,$$

$\implies \mathcal{R}(P) \perp \mathcal{N}(P)$. Thus $\mathcal{H}$ can be expressed as a direct sum of $\mathcal{R}(P)$ and $\mathcal{N}(P)$ and hence $\mathcal{R}(P) = (\mathcal{N}(P))^{\perp}$. Thus $\mathcal{R}(P)$ is closed.

(1) $\implies$ (2) Suppose $E = \mathcal{R}(P)$, then by the assumption $\mathcal{H} = E \bigoplus E^{\perp}$, with $\mathcal{N}(P) = E^{\perp}$. Since $P \neq O \implies E \neq \{0\}$.
Thus there is $x \in E$ with $\|x\| = 1$. Then, by $P^2 = P$ and $x \in \mathcal{R}(P)$, $P(x) = x$. In other words, $\|Px\| = \|x\| = 1$. So $\|P\| \geq 1$ because $\|P\| = \sup \frac{\|Px\|}{\|x\|}$.
On the other hand, invoking Pythagoras we see, if $x \in \mathcal{H}$, then $x = y + z$ with $y \in E$ and $z \in E^{\perp}$, then

$$\begin{aligned}\|Px\|^2 &= \|y\|^2 \\ &= \|x\|^2 - \|z\|^2 \\ &\leq \|x\|^2\end{aligned}$$

We see $\|P\| \leq 1$ and conclude $\|P\| = 1$.

(2) $\implies$ (1) Assume $\|P\| = 1$. let $x \in \mathcal{N}(P),\ y \in \mathcal{R}(P)$, then for $\lambda \in \mathbb{C}$

$$\begin{aligned}\|\lambda y\|^2 &= |\lambda|^2\|y\|^2\\ &= \|P(x+\lambda y)\|^2\\ &\leq \|x+\lambda y\|^2\\ &\leq \|x\|^2 + 2Re[\ \overline{\lambda}\,\langle x,y\rangle\ \ ] + \lambda^2\|y\|^2\end{aligned}$$

Subtracting $|\lambda|^2\|y\|^2$ from both side gives

$$\|x\|^2 + 2Re[\ \overline{\lambda}\,\langle x,y\rangle\ \ ] \geq 0\ ,$$

Now choosing $\lambda = t\langle x,y\rangle$ gives for each $t \in \mathbb{R}$, gives that

$$\|x\|^2 + 2t|\langle x,y\rangle|^2 \geq 0, \quad \forall\ \ t \in \mathbb{R}$$

$\implies |\langle x,y\rangle| = 0$, then we conclude $\langle x,y\rangle = 0$.
So $\mathcal{N}(P) \perp \mathcal{R}(P)$. Also we know that $\mathcal{H} = \mathcal{N}(P) \bigoplus \mathcal{R}(P)$,
so it is orthogonal decomposition when $\|P\| = 1$.
Thus, P is orthogonal projection.