Lecture Notes from September 15, 2022

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Last Time

- Every bounded linear operator can be identified by it's quadratic form.
- Equivalent conditions of normal operators
- Equivalent conditions of P being an orthogonal projection.

Warm up:

2.51 Question. If $\mathcal{H}$ has finite dimension, $A : \mathcal{H} \to \mathcal{H}$ satisfies $A^*A = \mathrm{id}_{\mathcal{H}}$, then A is unitary.

Since $A^*A = \mathrm{id}_{\mathcal{H}}$, we have A^* is surjective. Hence using the previous lemma we have $\mathcal{N}(A) = \mathcal{R}(A^*)^{\perp} = 0$. So A is one-to-one and counting dimensions, by Rank-Nullity, A is onto. Also A being an isometry and surjective is an unitary.

Unfortunately, A might not be surjective if $\mathcal{H}$ has infinite dimensions.

3 Orthogonal Projections

We begin with the following theorem for equivalent conditions of Orthogonal projections.

3.1 Theorem. *Let $0 \neq P \in \mathcal{B}(\mathcal{H})$ be a projection. i.e., $P^2 = P$. Then the following are equivalent:*

(1) P is an orthogonal projection, so $\mathcal{N}(P) \perp [\mathcal{R}(P)$.

(2) $\|P\| = 1$

(3) $\langle Px, x \rangle \geq 0$ for each $x \in \mathcal{H}$.

(4) P is hermition. That is, $P^* = P$.

(5) P is normal.

Proof. We have already shown that (1) $\implies$ (2). Now to show (2) $\implies$ (1).

Let $x \in \mathcal{N}(P), y \in \mathcal{R}(P)$, then for $\lambda \in \mathbb{C}$

$$\begin{aligned} \|\lambda y\|^2 = |\lambda|^2\|y\|^2 &= \|P(x + \lambda y\| \\ &\leq \|x + \lambda y\|^2 \\ &= \|x\|^2 + 2\mathrm{Re}[\overline{\lambda}\langle x, y\rangle] + |\lambda|^2\|y\|^2 \end{aligned}$$

□

Subtracting $\|\lambda y\|^2$ from both sides, we get

$$\|x\|^2 + 2\mathrm{Re}[\overline{\lambda}\langle x, y\rangle] \geq 0$$

for any $\lambda \in \mathbb{C}$

Setting $\lambda = t\langle x, y\rangle$ gives for each $t \in \mathbb{R}$, $\|x\|^2 + 2t|\langle x.y\rangle|^2 \geq 0$. So conclude, for this to hold for each t, $\langle x, y\rangle = 0$.

Hence, $\mathcal{N}(P) \perp \mathcal{R}(P)$ and also since $\mathcal{N}(P)$ and $\mathcal{R}(P)$ are closed subspaces of $\mathcal{H}$, so $\mathcal{H} = \mathcal{N}(P) \bigoplus \mathcal{R}(P)$, so P is an orthogonal projection..

So we have show (1) $\iff$ (2).

Next, we prove (1) $\implies$ (3), this follows from

$$\begin{aligned} \langle Px, x\rangle = \langle Px, x - px + px\rangle &= \langle Px, (I - P)x + Px\rangle \\ &= \langle Px, Px\rangle \\ &= \|Px\|^2 \geq 0 \end{aligned}$$

(3) $\implies$ (4), Since the quadratic form of P is non-negative, we have $x \mapsto \langle Px, x\rangle \in \mathbb{R}$ for each $x \in \mathcal{H}$, P is Hermition by our theorem on Sesquilinear/ Quadratic forms(Theorem 1.48(6)).

(1) $\implies$ (5) We recall $P = P^*$ implies $PP^* = P.P = P^*P$. So P is normal.

It is left to show (5) $\implies$ (1). Let P be a projection and P is normal. Then for each $x \in \mathcal{H}$, by Theorem. , we have

$$\|Px\| = \|P^*x\|$$

Hence, $Px = 0 \iff P^*x = 0$, and we get $\mathcal{N}(P) = \mathcal{N}(P^*)$.

By orthogonality relation, $\mathcal{N}(P^*) = (\mathcal{R}((P^*)^*))^\perp = [\mathcal{R}(P)]^\perp$

3.2 Examples. We consider an example of an orthogonal projection that maps onto the range of an isometry.

Let $S : l^2 \to l^2$, defined by $(Sx)_j = x_{j+1}$.

Then for $x, y \in l^2$ consider,

$$\langle x, S^*y\rangle = \langle Sx, y\rangle = \sum_{j=1}^{\infty} (Sx)_j \overline{(y)_j} = \sum_{j=1}^{\infty} (x)_{j+1} \overline{(y)_j}$$
$$= \langle (x_1, x_2, \cdots), (0, y_1, y_2, \cdots)\rangle$$
$$= \langle x, S^*y\rangle$$

which is true for any $x, y \in \mathcal{H}$. Hence we have, $(S^*x)_j = \begin{cases} 0 & \text{if } j = 1 \\ x_{j-1} & \text{if } j \geq 2 \end{cases}$,

Because of this, we see $SS^* = id_{\mathcal{H}}$. Hence S^* is an isometrty, and $(S^*x)_j = \begin{cases} 0 & \text{if } j = 1 \\ x_j & \text{if } j \geq 2 \end{cases}$ projects orthogonally onto the range of S^*. Also, since $SS^* \neq S^*S$ and hence S not normal.

4 Spectral Theory

Warm up: The usual route to spectrum is given by the resolvent of the bounded linear operator A. Consider the operator $T_z : A - z\mathrm{id}_{\mathcal{H}}$ and ask if this is bounded and invertible. Then we call $R_z = (A - z\mathrm{id}_{\mathcal{H}})^{-1}$ the resolvent of A which is usually the central discussion to learn about the spectrum.

However, we are going to follow a different route here.
The main goal here is to understand the behavior of normal operators specifically unitary and hermition ones. Representation theory offers a good framework for generating insights.

4.1 Question. What is a represenation?

A represenation is a map from some structured set to operators on a hilbert space.

We introduce a natural, minimal structure.

4.2 Definition. A pair $(S, *)$ of a semigroup with an involutive anti-automorphism $s \to s^*$ is called **involutive semi-group**.

- The anti-automorphism gives $(st)^* = t^*s^*$, reverses the order of compositions/multiplication.
- If 1 is a unit, then $1^* = (1.1)^* = 1^*1^* \implies 1 = 1^*$ (since 1 being a unit is invertible, hence $1^* = 1^{-1} \implies 1^*$ is invertible. Since we have $1^* = 1^*1^*$, so right multiplying with 1^{*-1}, we get $1 = 1^*.1^{*-1} = 1^*.1^*.1^{*-1} = 1^*.1 = 1^*$).

4.3 Definition. Elements in $S_h = s : s = s^*$ are called **hermition** and $S_u = s : ss^* = s^*s = 1$ are called **unitaries**. The set S_u along with $*$ forms a group, called the **unitary group**.

4.4 Examples.

1. If S is an abelian semigroup, then (S, id_s) is a involutive semigroups.
2. If G is a group, and we let $g* = g^{-1}$(as $(gh)^{-1} = h^{-1}g^{-1}$, then $(G, *)$ is an involutive group.
3. $\mathcal{B}(\mathcal{H})$ with $A \to A^*$ is an involutive semigroup. And $\mathcal{B}(\mathcal{H}))_u$ is the set of all unitaries.
4. The multiplicative semi-group $\mathbb{C}$ is an involutive semi-group with $z^* = \overline{z}$. This is $\mathcal{B}(\mathbb{C}^1)$.
5. If X is $\underline{\text{a set}}$, then $\mathbb{C}^*$ is an involutive semigroup(under pointwise multiplication) with $f^*(x) = \overline{f(x)}$ and $(fg)(x) = f(x)g(x)$.
   A function f is hermition if it is real valued(since f is hermition $\iff f^* = f \iff \overline{f(x)} = f(x)$).