Lecture Notes from September 15, 2022

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Last time

- Characterization of unitaries
- Characterization of isometries (unitaries + a condition)
- Geometric characterization of normal operators
- Analogue of rank-nullity for $A \in \mathbb{B}(\mathcal{H})$
- Characterization of orthogonal projections (proof in today's class)

Warm up:

1.47 Question. If $\mathcal{H}$ is a finite-dimensional Hilbert space and $A : \mathcal{H} \to \mathcal{H}$ a linear map satisfying $A^*A = \mathrm{id}_{\mathcal{H}}$. Then A is unitary.

We will see that A is onto, then (onto + isometry) $\implies$ unitary. A^* is onto since $A^*A = \mathrm{id}_{\mathcal{H}}$ so given any $x \in \mathcal{H}$, $A^*Ax = x$ hence there exists $y = Ax \in \mathcal{H}$ such that $A^*y = x$. We also know that $\mathcal{N}(A) = \mathcal{R}(A)^{\perp} = \{0\}$ and so A is one-one. Next, using Rank-Nullity we have that $\dim \mathcal{H} = \dim \mathcal{N}(A) + \dim \mathcal{R}(A) = \dim \mathcal{R}(A)$, $\mathcal{H} = \mathcal{R}(A)$ and so A is onto and hence A is unitary.

We continue the proof of the theorem from last time that characterized orthogonal projections. Let us recall the theorem

1.48 Theorem. *Suppose $P \in \mathbb{B}(\mathcal{H})$ be a non-zero projection, i.e., $P^2 = P$, then TFAE:*

1. *P is an orthogonal projection, so $\mathcal{N}(P) \perp \mathcal{R}(P)$.*
2. *$\|P\| = 1$.*
3. *$\langle Px, x\rangle$.*
4. *$P = P^*$.*
5. *P is normal.*

Last time we saw that given any projection (not necessarily orthogonal), we always have the direct sum $\mathcal{H} = \mathcal{N}(P) + \mathcal{R}(P)$, $x = Px + (I-P)x$. Here $\mathcal{N}(P) = P^{-1}(\{0\})$ and $\mathcal{R}(P) = (I-P)^{-1}(\{0\})$ (since $x \in (I-P)^{-1}(\{0\}) \iff (I-P)x = 0 \iff Px = x \iff x \in \mathcal{R}(P)$) are both closed subspaces since P is bounded. We prove the following chain of equivalences: ($1. \iff 2.$ and $1. \implies 3. \implies 4. \implies 5. \implies 1.$) We already proved that ($1. \implies 2.$). Now we prove the rest of the equivalences.

Proof. ($2. \implies 1.$) We have $\|P\| = 1$. Let $x \in \mathcal{N}(P), y \in \mathcal{R}(P)$. For any $\lambda \in \mathbb{C}$,

$$\begin{aligned}
\|\lambda y\|^2 &= |\lambda| \|y\|^2 \\
&= \|P(x+\lambda y)\|^2 \quad (since\ Px = 0\ and\ P\lambda y = \lambda P y = \lambda y) \\
&\leq \|x + \lambda y\|^2 \quad (since\ \|P\| = 1 \implies P\ is\ contractive) \\
&= \langle x+\lambda y, x+\lambda y\rangle \\
&= \|x\|^2 + 2\mathrm{Re}(\overline{\lambda})\langle x, y\rangle + \|\lambda y\|^2
\end{aligned}$$

Then $\|\lambda y\|^2 \leq \|x\|^2 + 2\mathrm{Re}(\overline{\lambda})\langle x, y\rangle + \|\lambda y\|^2$ which gives $\|x\|^2 + 2\mathrm{Re}(\overline{\lambda})\langle x, y\rangle \geq 0$ for all $\lambda \in \mathbb{C}$. Set $\lambda = t\langle x, y\rangle$ for $t \in \mathbb{R}$ so that $\overline{\lambda} = t\langle y, x\rangle$. Then

$$\|x\|^2 + 2\mathrm{Re}(t|\langle x, y\rangle|^2 \geq 0.$$

This inequality holds for all $t \in \mathbb{R}$, hence if we choose t to be a small enough negative number ($t < \frac{-\|x\|^2}{2|\langle x,y\rangle|^2}$) then the inequality does not hold unless $\langle x, y\rangle = 0$. Since we began with $x \in \mathcal{N}(P), y \in \mathcal{R}(P)$, we have $\mathcal{N}(P) \perp \mathcal{R}(P)$.

($1. \implies 3.$)

$$\begin{aligned}
\langle Px, x\rangle &= \langle Px, x - Px + Px\rangle \\
&= \langle Px, x - Px\rangle + \langle Px, Px\rangle \\
&= \langle Px, (I-P)x\rangle + \|Px\|^2 \qquad = \|Px\|^2 \text{ since } \mathcal{N}(P) = \mathcal{R}(I-P) \perp \mathcal{R}(P)
\end{aligned}$$

Thus, $\langle Px, x\rangle = \|Px\|^2 \geq 0$

($3. \implies 4.$) We saw earlier, in the theorem on sesquilinear and quadratic forms, that an operator P is Hermitian if, and only if $\forall x \in \mathcal{H}$, $x \mapsto \langle Px, x\rangle \in \mathbb{R}$ which holds since $\langle Px, x\rangle \geq 0$.

($4. \implies 5.$) P is Hermitian $P = P^* \implies$ P is normal, $PP^* = P^*P = P^2$.

($5. \implies 1.$) If P is normal, then for each $x \in \mathcal{H}$, $\|Px\| = \|P^*x\|$ hence $Px = o \iff P^*x = 0$. We thus get that

$$\mathcal{N}(P) = \mathcal{N}(P^*) = \mathcal{R}(P^{**})^{\perp} = \mathcal{R}(P)^{\perp}.$$

Hence $\mathcal{N}(P) \perp \mathcal{R}(P)$. □

A good way of summarizing the properties of an orthogonal projection is $P = PP^*$ since this implies $P = P^*$ and $P = P^2$.

1.49 Examples (The left shift operator). Let

$$S : \ell^2 \longrightarrow \ell^2 \quad (Sx)_j = x_{j+1}.$$

It takes the element $x = (x_1, x_2, \cdots) \in \ell^2$ to $(x_2, x_3, \cdots) \in \ell^2$. ℓ^2 is spanned by the orthonormal basis $\{\delta_s : s \in \mathbb{N}\}$ and $\langle Sx, x\rangle = \langle x, S^*x\rangle$ for all $x \in \ell^2$. For the basis vectors, we have for $s \geq 2$

$$\begin{aligned}
\langle S\delta_s, \delta_t\rangle &= \langle \delta_s, S^*\delta_t\rangle \\
\langle \delta_{s+1}, \delta_t\rangle &= \langle \delta_s, S^*\delta_t\rangle \\
\langle \delta_{s+1}, \delta_t\rangle &= 1 \text{ for } s+1 = t \text{ and } 0 \text{ otherwise} \\
\text{Thus } \langle \delta_s, S^*\delta_t\rangle &= 1 \text{ for } s+1 = t \text{ and } 0 \text{ otherwise}, \\
(S^*\delta_{s+1})_s = 1 &\implies S^*\delta_s = \delta_{s-1}.
\end{aligned}$$

Note that $\langle S\delta_1, \delta_t\rangle = 0 = \langle \delta_s, S^*\delta_t\rangle$ for all t thus $S^*\delta_1 = 0$. By extending linearly, we see that the adjoint is given by the right shift operator

$$S^* : \ell^2 \longrightarrow \ell^2 \quad (S^*x)_1 = 0;\ (S^*x)_j = x_{j-1} \text{ for } j \geq 2$$

It takes the element $(x_1, x_2, \cdots) \in \ell^2$ to $(0, x_1, x_2, \cdots) \in \ell^2$. The map S^* is an isometry ($S^*S = id$) and is not onto since (thus not unitary) the element $(x, 0, 0, \cdots$ has no preimage under S^*. The map S^*S given by $(S^*Sx)_1 = 0$; $(S^*Sx)_j = x_j$ for $j \geq 2$ projects orthogonally onto $\mathcal{R}(S^*)$.

2 Spectral Theory

Recall, from linear algebra, the concept of eigenvalues and eigenvectors. These gave us a lot of information about matrices (or operators on finite dimensional vector spaces). In general, for studying operators on Hilbert spaces, a generalized notion called the *spectrum* is introduced and studied. It is defined as follows. Given an operator A, consider $T_z = A - z\mathrm{Id}_{\mathcal{H}}$ for $z \in \mathbb{C}$ and ask if T_z has a bouned inverse. The resolvent is the set of $\{z \in \mathbb{C} : T_z\ invertible\}$ and the spectrum is the complement of the resolvent. If T_z is invertible, the inverse is given by a polynomial in powers of A. Thus the Neumann series $\sum_i \lambda_i A^i$ [1]associated with A, are studied to understand the resolvent and spectrum. However, we will take a slightly different approach in this course to introduce these notions. The main goal here is to understand the behaviour of normal operators, especially unitary and Hermitian ones. Representation theory offers a good framework for generating insight. A representation is a map from some 'structured space' to operators on a Hilbert space. We start with a definition of involutive semigroups.

2.1 Definition (Involutive semigroup)**.** A pair $(\pi : S, *)$ of a semigroup S with an involutive anti-automorphism $s \mapsto s^*$ is called an *involutive semigroup*.

Note:

- The anti-automorphism gives $(st)^* = t^*s^*$, i.e., it reverses the order of the semigroup operation.

[1] `https://en.wikipedia.org/wiki/Neumann_series`

- If 1 is a unit, then $1s = s\ \forall s \in S$. In particular, $11^* = 1^*$. Thus, $(11^*)^* = (1^*)^*$ gives $11^* = 1$ and so $1^* = 1$.

2.2 Definition (Hermitian and unitaries)**.** Elements in $S_h = \{s \in S : s = s^*\}$ are called *hermitian*, and elements in $S_u = \{s \in S : ss^* = s^*s = 1\}$ are called *unitaries*. S_u forms a group, the unitary group of S.

2.3 Examples. 1. If S is an abelian semigroup and $s^* = s$, $s \in S$, then $(S, *)$ is an involutive semigroup.

2. If G is a group, and $g^* = g^{-1}$, $g \in G$ then $(G, *)$ is an involutive semigroup.

3. $\mathbb{B}(\mathcal{H})$ with the adjoint operation $A \mapsto A^*$, $A \in \mathbb{B}(\mathcal{H})$ is an involutive semigroup.

4. The multiplicative semigroup $\mathbb{C} \setminus \{0\}$ with $z^* = \bar{z}$, $z \in \mathbb{C}$ is an involutive semigroup.

5. If X is any set then define $\mathbb{C}^X := \{f : X \to \mathbb{C}\}$, the set of all maps from X to $\mathbb{C}$ with semigroup operation $(fg)(x) = f(x)g(x)$ and involution $f^*(x) = \overline{f(x)}$, $x \in X$. $(\mathbb{C}^X, *)$ is an involutive semigroup.

   - The Hermitian elements are precisely those functions satisfying $f(x) = \overline{f(x)}\ \ \forall x \in X$, i.e., the real-valued functions ($\mathbb{C}^X_h = \mathbb{R}^X$).
   - The identity 1 satisfies $f(x)1(x) = f(x)\ \ \forall x \in X$. $1(x) = 1 \forall x \in X$.
   - The unitaries are $\mathbb{C}^X_u = \{f : X \to \mathbb{C} : |f(x)| = 1\ \ \forall x \in X\}$. orthogonal projections given by $f(x) = |f(x)|^2\ \ \forall x \in X$ are precisely the functions with values 0 and 1.