Lecture Notes from September 20, 2022

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Last Time

- Characterization of orthogonal projection.
- Isometries v/s orthogonal projections.
- Semigroups

Warm up: Let X be a set, consider $\mathbb{C}^X \equiv \{f; f : X \to \mathbb{C}\}$ such that $(fg)(x) = f(x)g(x), f^*(x) = \overline{f(x)}$.
If f is an orthogonal projection, then $f^*f = f \iff \bar{f}f = f \iff |f|^2 = f \iff$ f has values 0 or 1.

4.5 Question. What are unitaries in $\mathbb{C}^x$?

The identity of $\mathbb{C}^X$ is $e(x) = 1$.Unitary group of the semigroup is $\mathbb{C}^X_u = \{f : X \to \mathbb{C} : \forall x \in X, |f(x)| = 1\}$

4.6 Definition. A representation of an involutive semigroup S is a homomorphism $\pi : S \to B(\mathcal{H})$ where $\mathcal{H}$ is a complex Hilbert space, and for which $\pi(s^*) = (\pi(s))^*$ for each $s \in S$. We also write $(\pi, \mathcal{H})$ if we need to keep track of $\mathcal{H}$.
If G is a group with $g^* = g^{-1}$ and $\pi(1) = id_{\mathcal{H}}$ then the representation is called unitary.

4.7 Remark.

1. From $g^*g = gg^* = 1$, we get $\pi(g) = B(\mathcal{H})_u$, so a unitary (group) representation maps into the unitary group of Hilbert space (since, $\pi(s^*s) = \pi(s^{-1}s = \pi(1) = id_{\mathcal{H}}$ and $\pi(s^*)\pi(s) = \pi(s^{-1}\pi(s) = (\pi(s))^{-1}\pi(s) = id_{\mathcal{H}})$.
2. In general, if 1 is a unit of S and π is a representation, then we only know that $\pi(1) = (\pi(1)(\pi(1))^*$, $\pi(1)$ is an orthogonal projection. So $\pi(1)$ is an orthogonal projection based on the characterization of orthogonal projection in class.

4.8 Definition. Two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H})$ of an involutive semigroup are called equivalent if there is a unitary $u \in B(\mathcal{H}, \mathcal{H}')$ and for each $s \in S, u \circ \pi(s) = \pi'(s) \circ u$. In this context, u is called interwining operator. (Since, $u^*u \circ \pi(s) = u^*\pi'(s)u \implies \pi(s) = u^*\pi'(s) \circ u$. Conversely, $u \circ \pi(s) \circ u^* = \pi'(s) \circ u \circ u^* \implies \pi^*(s) = u \circ \pi(s) \circ u^*$). This shows that we can learn all about π from π' and vice versa.

4.9 Remarks. 1. Let $S = Z$ (additive) and $s^* = s$, and let $(\pi, \mathcal{H})$ be the unitary representation of S, so $\pi(1) = u \in (B(\mathcal{H}))_u$ and then $\pi(0) = u^0 \equiv id_{\mathcal{H}}$, $u^{-n} \equiv (u^{-1})^n$.
Conversely, given the unitary operator u on $\mathcal{H}$, $\pi(n) = u^n$ defines a representation of Z on $\mathcal{H}$.
Studying U and studying the associated representation concerns the same information.

2. Let $S = (\mathbb{N}_0, +)$ such that $S = S^*$. If $(\pi, \mathcal{H})$ is a representation of S, then $A = \pi(1)$, then $\pi(n) = A^n$ and $A = A^*$. Studying representations of S is equivalent to studying each $A \in (B(\mathcal{H}))_h$

3. Let $(\mathbb{N}_0 \times \mathbb{N}_0, +)$ with $((n, m))^* = (m, n)$. Then this is an abelian involutive semigroup. If $(\pi, \mathcal{H})$ is a representation, $A = \pi(1.0)$ defines π. Assuming $\pi(0, 0) = id\mathcal{H}$, we have, $\pi(0, 1) = \pi((1, 0))^* = A^*$, and $AA^* = \pi(1, 0)\pi(0, 1) = \pi(1, 1) = \pi(0, 1)\pi(1, 0) = A^*A$. So, A is normal, and for each $n, m \in \mathbb{N}_0$, $\pi(n, m) = A^n(A^*)^m$

We have thus converted between studying unitary, hermitian and normal operators and study of representations of semigroups.

4.10 Question. Give a representation, can it be reduced to fundamental ingredients/parts?

4.11 Definition. 1. A representation of an involutive semigroup is called non-degenerate if $\pi(S)\mathcal{H} = \{\pi(s)v : s \in S, v \in \mathcal{H}\}$ is total or equivalently $\pi(S)\mathcal{H}$ is dense in $\mathcal{H}$. This is the case if S has a unit 1 and $\pi(1) = id_{\mathcal{H}}$(Since $\pi(1)$ is an orthogonal projection, in general it projects onto a closed subspace of $\mathcal{H}$. Therefore, $\pi(S)\mathcal{H}$ is the subset of the closed subspace which may or may not be in $\mathcal{H}$).

2. A representation $(\pi, \mathcal{H})$ of an involutive semigroup is called cyclic, if there is $v \in \mathcal{H}$ such that $\pi(S)v = \{\pi(s)v : s \in S\}$ is total.

3. A representation $(\pi, \mathcal{H})$ of an involutive semigroup is called irreducible if 0 and $\mathcal{H}$ are the only closed subspaces that are invariant under $\pi(S)$.

4.12 Lemma. *Let* $(\pi, \mathcal{H})$ *be a representation of an involutive semigroup* S , $E \subset \mathcal{H}$ *be a subspace, and* P_E *be the orthogonal projection onto* E*, then the following are equivalent:*

1. E *is invariant under* S.

2. $E^\perp$ *is invariant under* S.

3. $P_E\pi(S) = \pi(S)P_E$

Proof. (1) $\implies$ (2)

By $(\pi(s))^* = \pi(s^*)$ We know E is invariant under $\pi(s)$ if and only if $E^\perp$ is invariant under $\pi(s^*)$. Hence, E is invariant under $\pi(S)$ if and only if $E^\perp$ is invariant under $\pi(S)$.
(2) $\implies$ (3)
Let $v \in \mathcal{H}, v_E = P_E v$. For $s \in S$

$$\begin{aligned}
P_E\pi(s)v &= P_E\pi(s)(\underbrace{v_E}_{\in E} + \underbrace{v - v_E}_{\in E^\perp}) \\
&= \underbrace{P_E(\pi(s)v_E}_{\in E} + \underbrace{\pi(s)(v - c_E)}_{\in E^\perp} \\
&= \underbrace{P_E\pi(s)v_E}_{\in E} \\
&= \pi(s)P_E v_E \\
&= \pi(s)P_E(v_E + v - v_E)
\end{aligned}$$

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We conclude $P_E\pi(s) = \pi(s)P_E$ for each $s \in S$.

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