Lecture Notes from September 20, 2022

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Last time

- Characterization of orthogonal projections.
- Isometries versus orthogonal projections.
- Semigroups.

Warm up:

Let X be a set. Last class, we considered the space

$$\mathbb{C}^X \equiv \{f : X \to \mathbb{C}\},$$

as a semigroup, where the operation is pointwise multiplication: $(fg)(x) = f(x)g(x)$, and the anti-automorphism is given by the complex conjugate: $f^*(x) = \overline{f(x)}$. We might consider the following:

1.6 Question. What are the unitaries in $\mathbb{C}^X$? What about the orthogonal projections?

Well, the identity in $\mathbb{C}^X$ is $e(x) \equiv 1$. Hence the unitary group of the semigroup is

$$(\mathbb{C}^X)_u = \{f : X \to \mathbb{C} : \text{for all } x \in X,\ |f(x)| = 1\}$$

since $f(x)f^*(x) = f(x)\overline{f(x)} = |f(x)|^2$. As for the orthogonal projections, they are the 0 or 1 valued functions as they satisfy $f(x)\overline{f(x)} = f(x)$ from our characterization of orthogonal projections.

A bit of wisdom before we continue:

If anyone asks you in anything in math, you just say zero. And if they say no, then you say one.
- Professor Bodmann

1.7 Definition. A *representation* of an involutive semigroup S is a homomorphism $\pi : S \to B(\mathcal{H})$, that is,

$$\pi(ab) = \pi(a)\pi(b)$$

for all $a, b \in S$, where $\mathcal{H}$ is the complex Hilbert space, and for which

$$\pi(s^*) = (\pi(s))^* \quad \text{for each } s \in S.$$

- We write the representation as a pair $(\pi, \mathcal{H})$ if we need to keep track of $\mathcal{H}$.
- If $S = G$ is a group with $g^* = g^{-1}$ and $\pi(1) = \mathrm{id}_{\mathcal{H}}$, then the representation is called *unitary*.

1.8 Question. Why is it a 'good' idea to call such representations unitaries?

From $g^*g = gg^* = 1$, we have $\pi(g) \in (B(\mathcal{H}))_u$, so a unitary (group) representation maps into the unitary group of a Hilbert space.
• In general, if 1 is a unit of S and π are representation, then we only know $\pi(1) = \pi(1)(\pi(1))^*$, which implies that $\pi(1)$ is an orthogonal projection.

1.9 Definition. Two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ of an involutive semigroup S are called equivalent if there exists a unitary $u \in B(\mathcal{H}, \mathcal{H}')$ and for each $s \in S$,

$$u \circ \pi(s) = \pi'(s) \circ u.$$

• In this context, u is called an *intertwining operator*.

1.10 Exercise. Show that the above definition induces an equivalence relation $\sim$ on representations of involutive semigroups.

Proof sketch. Reflexive. $\pi \sim \pi$ since for each unitary $u \in B(\mathcal{H})$ and $s \in S$, we have

$$u \circ \pi(s) \circ u^{-1} = \pi(s) \quad \Rightarrow \quad u \circ \pi(s) = \pi(s) \circ u.$$

Symmetric. Suppose $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are two representations of an involutive semigroup S such that $\pi \sim \pi'$. Then there exists a unitary $u \in B(\mathcal{H}, \mathcal{H}')$ such that

$$\begin{aligned} & u \circ \pi(s) = \pi'(s) \circ u \\ \iff & \pi(s) = u^{-1} \circ \pi'(s) \circ u \\ \iff & \pi(s) \circ u^{-1} = u^{-1} \circ \pi'(s), \end{aligned}$$

hence, $\pi' \sim \pi$.
Transitive. We leave it to the reader. □

1.11 Remark. (a) Let $S = \mathbb{Z}$ be the group of integers under addition, with $s^* = -s$, and let $(\pi, \mathcal{H})$ be a unitary representation of S. Then $\pi(1)$ is a unitary $\pi(1) = u \in (B(\mathcal{H}))_u$, hence

$$\pi(n) = \pi(1)^n = u^n \quad \text{for each } n \in \mathbb{Z}$$

with $\pi(0) = u^0 = \mathrm{id}_{\mathcal{H}}$. Also,

$$u^{-n} \equiv (u^{-1})^n = (u^*)^n \quad \text{for each } n \in \mathbb{Z}.$$

Conversely, given a unitary u on $\mathcal{H}$, $\pi(1) = u^n$ defines a representation of $\mathbb{Z}$ on $\mathcal{H}$. We see that studying u and studying the associated representations concern the same information.

(b) Let $S = (\mathbb{N}_0, +)$, $s^* = s$. If $(\pi, \mathcal{H})$ is a representation of S and $A = \pi(1)$, then for $n \in \mathbb{N}$ we have

$$\pi(n) = A^n \quad \text{and} \quad A = A^*.$$

We observe that studying representations of S is equivalent to studying each $A \in (B(\mathcal{H}))_h$.

(c) Let $S = (\mathbb{N}_0 \times \mathbb{N}_0, +)$ with $(n, m)^* = (m, n)$. If $(\pi, \mathcal{H})$ is a representation of S, then we claim that $A = \pi(1, 0)$ determines π, assuming $\pi(0, 0) = \mathrm{id}_{\mathcal{H}}$. We have

$$\begin{aligned} \pi(0,1) &= \pi((1,0)^*) \\ &= (\pi(1,0))^* = A^*, \end{aligned}$$

and hence

$$\begin{aligned} AA^* &= \pi(1,0)\pi(0,1) = \pi(1,1) \\ &= \pi(0,1)\pi(1,0) = A^*A, \end{aligned}$$

which implies that A is normal, and for all $m, n \in \mathbb{N}_0$, $\pi(m, n) = A^m(A^*)^n$. This shows the claim. Conversely, we see each normal operator defines a representation of this group.

We have thus converted between studying unitary, Hermitian, and normal operators and the study of representations of specific semigroups.

1.12 Question. Given a representation, can it be reduced to the fundamental ingredients/parts?

1.13 Definition. (a) A representation $(\pi, \mathcal{H})$ of an involutive semigroup is called *non-degenerate* if

$$\pi(S)\mathcal{H} = \{\pi(s)v : s \in S,\ v \in \mathcal{H}\}$$

is total, or equivalently, $\pi(S)\mathcal{H}$ is dense in $\mathcal{H}$.

(This is the case if S has a unit 1 and $\pi(1) = \mathrm{id}_{\mathcal{H}}$.)

(b) A representation $(\pi, \mathcal{H})$ of an involutive semigroup is called *cyclic* if there is $v \in \mathcal{H}$ such that

$$\pi(S)v = \{\pi(s)v : s \in S\}$$

is total.

(c) A representation $(\pi, \mathcal{H})$ of an involutive semigroup is called *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces that are invariant under $\pi(S)$.

1.14 Remark. Irreducible representations are always cyclic. We will show this later.

1.15 Lemma. *Let $(\pi, \mathcal{H})$ be a representation of an involutive semigroup S, $E \subset H$ a closed subspace, and P_E an orthogonal projection onto E. The following are equivalent:*

1. *E is invariant under π.*
2. *$E^\perp$ is invariant under π.*
3. *$P_E\pi(S) = \pi(S)P_E$.*

Proof strategy: Show (1) and (2) are equivalent, then use them to show that (3) holds, and finally show that (3) implies (1).

Proof. (1) $\iff$ (2). We start by observing that $\pi(s)^* = \pi(s^*)$ implies E is invariant under $\pi(s)$ if and only if $E^\perp$ is invariant under $\pi(s^*)$. Hence, E is invariant under $\pi(s)$ if and only if $E^\perp$ is invariant under $\pi(s)$. Now assuming that E is invariant under π, E is invariant under $\pi(s)$ for all $s \in S$, and equivalently, this holds true for all $s^* \in S$. This is equivalent to $E^\perp$ being invariant under $(\pi(s^*))^* = \pi(s)^{**} = \pi(s)$ for all $s^* \in S$, which is equivalent to $E^\perp$ invariant under $\pi(s)$ for all $s \in S$. This proves that E is invariant under π if and only if $E^\perp$ is invariant under π.

(1) and (2) $\Rightarrow$ (3). Let $v \in \mathcal{H}$ and set $v_E = P_E v$. For $s \in S$,

$$\begin{aligned} P_E\pi(s)v &= P_E\pi(s)(\underbrace{v_E}_{\in E} + \underbrace{v - v_E}_{\in E^\perp}) \\ &= P_E(\pi(s)v_E + \pi(s)(v - v_E)) \\ &= P_E\pi(s)v_E \\ &= \pi(s)P_E v. \end{aligned}$$

We conclude $P_E\pi(s) = \pi(s)P_E$ for all $s \in S$. We delay the rest of the proof until the next lecture. □