Operators on Hilbert Spaces

Lecture Notes from September 22, 2022

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Warm up

- Is it necessary for the involutive semigroup representation to "respect" the involution property $((\pi(s^*) = (\pi(s))^*)$ to satisfy the equivalent properties in the lemma we did in the last lecture ?

Solution: Consider the semigroup $S = \mathbb{N}$, with addition. Let π be a semigroup representation given $\pi(1) = S^*, S^* : l^2 \mapsto l^2$ as

$$(S^*x)_j = \left\{ \begin{array}{ll} 0; & \text{if} j = 1 \\ x_{j-1}; & \text{if} j \geq 2 \end{array} \right\} \tag{1}$$

Notice that $(\pi(1))^* = (S^*)^* = S \neq S^* = \pi(1) = \pi(1^*)$. Further, observe that for $k \in \mathbb{N}$, π has invariant subspaces of the form,

$$V_k = \{x \in l^2; x_j = 0, 1 \leq x \leq k\}$$

$$V_k = \overline{span\{e_j; j \geq k+1\}}$$

However, we note that if $k \geq 2$, P_{V_k} being the orthogonal projection onto V_k

$$P_{V_k}\pi(1)e_k = P_{V_k}S^*e_k = P_{V_k}e_{k+1} = e_{k+1} \neq \pi(1)P_{Vk}e_k = 0$$

This example shows that if the involutive semigroup representation does not "respect" the involution property, then property (1) and (2) $\not\Rightarrow$ property (3) in the lemma, we did in the last lecture.

Continuing from the last time

$(3) \implies (1)$

Assume (3) holds. Choose $v \in E$, $s \in S$ then

$$P_E\pi(s)v = \pi(s)P_Ev = \pi(s)v$$

which shows that $\pi(s)v \in E$.Therefore, E is invariant under π.

Next, we question how serious the requirement of non-degeneracy is.

1.39 Theorem. *If* (π, H)*is a representation of an involutive semigroup S then*

$$H_0 = \{v \in H; \forall s \in S, \pi(s)v = 0\}$$

is a closed subspace. The representation of S *on* $H_0^\perp$ *is non degenerate and* $H_0 = (\pi(S)H)^\perp$

Proof. Since

$$H_0 = \cap_{s\in S}\{v \in H; \pi(s)v = 0\}$$

where $\{v \in H; \pi(s)v = 0\}$ is a kernel of $\pi(s)$ and therefore closed subspace. Since H_0 is an arbitrary intersection of closed subspaces, hence H_0 is a closed subspace.
Using the relationship between orthogonal complement and adjoint, we get

$$H_0 = \cap_{s\in S}\mathcal{N}(\pi(s))$$

Since, $\mathcal{N}(\pi(s)) = (\mathcal{R}((\pi(s))^*))^\perp$ and $(\pi(s))^* = \pi(s^*)$, we get with $S = S^*$

$$H_0 = \cap_{s\in S}(\mathcal{R}(\pi(s)))^\perp$$

Using $\mathcal{R}(\pi(s)) = \pi(s)(H)$, we get

$$H_0 = (\cup_{s\in S}\pi(s)(H))^\perp = (\pi(S)(H))^\perp$$

Hence,

$$H_0^\perp = ((\pi(S)(H))^\perp)^\perp = \overline{span\pi(S)(H)}$$

By the previous lemma, since H_0 is invariant, so is $H_0^\perp$. Restricting π to $H_0^\perp$ gives a non-degenerate representation, because $\pi(S)H = \pi(S)(H_0^\perp)$ and hence span$\pi(S)(H_0^\perp)$ is dense in $H_0^\perp$. □

The preceding theorem gives us an example of showing how a Hilbert Space can be split into invariant subspaces under π. Conversely, we can combine representations.

1.40 Theorem. *Let* (π_j, H_j) *be a family of representations of an involutive semigroup. If*

$$\sup_{j\in J} \|\pi_j(s)\| < \infty$$

for each $s \in S$*, then for* $v = (v_j)_{j\in J} \in \oplus_{j\in J}H_j$

$$(\pi(s)v)_j = \pi_j(s)v_j$$

defines a representation of S *on* $H = \oplus_{j\in J}H_j$*.*

Proof. We verify that $\pi(s)$ maps H to itself. For $v = (v_j)_{j\in J} \in H$

$$\|\pi(s)v\|^2 = \sum_{j\in J} \|\pi_j(s)v_j\|^2 \leq \sup_{j'\in J} \|\pi_{j'}(s)\|^2 \sum_{j\in J} \|v_j\|^2 < \infty$$

where $\sum_{j\in J} \|v_j\|^2 = \|v\|^2$
Since each π_j being a homomorphism, then so is π.

We claim that

$$(\pi(s))^* = \pi(s^*)$$

For $v = (v_j)_{j\in J}$ and $w = (w_j)_{j\in J}$

$$\langle \pi(s)v, w\rangle = \sum_{j\in J} \langle \pi_j(s)v_j, w_j\rangle_{H_j} = \sum_{j\in J} \langle v_j, (\pi_j(s))^* w_j\rangle_{H_j}$$

The second equality follows by using the definition of the adjoint of $\pi_j(S)$. Now, $(\pi_j(s))^* = \pi_j(s^*)$. Using this and the definition of adjoint of $\pi(s)$, we get that,

$$\langle \pi(s)v, w\rangle = \langle v, (\pi(s))^* w\rangle = \langle v, \pi(s^*)w\rangle$$

Thus, $(\pi(s))^* = \pi(s^*)$

□

This representation that combines (π_j, H_j) is called the **direct sum representation** $\pi = \oplus_{j\in J}\pi_j$.

Next, we see how to decompose a non degenerate representation into its cyclic components.

1.41 Lemma. *Let (π, H) be a representation of an involutive semigroup and $v \in H$ then π is cyclic when restricted to $\overline{span\{\pi(S)v\}}$. If π is a non-degenerate then $v \in \overline{span\{\pi(S)v\}}$.*

Proof. Let $H_1 = \overline{span\{\pi(S)v\}}$. Since, $\pi(s) \in \mathcal{B}(H)$ for all $s \in S$, then $\pi(s)$ is continuous and also π is an homomorphism, which gives us that H_1 is invariant under $\pi(S)$. Therefore,- we get $H = H_0 \oplus H_1$ with $H_0 = H_1^\perp$. We write $v = v_0 + v_1$, $v_0 \in H_0$, $v_1 \in H_1$. Let $s \in S$. Since H_1 is invariant under $\pi(S)$, by the equivalence property in a lemma in the last lecture, we get $H_1^\perp$ is invariant under $\pi(S)$ i.e. H_0 is invariant under $\pi(S)$, therefore $\pi(s)v_0 \in H_0$.

$$\pi(s)v_0 = \pi(s)(v - v_1) = \pi(s)v - \pi(s)v_1 \in H_1$$

because $\pi(s)v, \pi(s)v_1 \in H_1$ and H_1 is a closed subspace. Hence, $\pi(s)v_0 = \{0\}$. Thus $\pi(S)v_1 = \pi(S)v$ and v_1 is a cyclic vector for H_1. If the representation is non-degenerate then $\overline{span\{\pi(S)\}(H)} = H$, so

$$v_0 \in \cap_{s\in S}\mathcal{N}(\pi(s)) = (\cup_{s\in S}\pi(s)(H))^\perp = \{0\},$$

and thus $v = v_1$, which implies $v \in H_1$. □