Lecture Notes from 22 September 2022

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Last time

- Properties of Representations
  - non-degeneracy
  - irreducibility
  - cyclicity
- Invariant Subspaces and Projections

Warm up:

1.2 Question. Is the involutive structure necessary for us to decompose Hilpert spaces into direct sums of invariant subspaces?

1.3 Answer. Yes, we can construct a counter example when the representation does not respect the involutive structure.

Consider the semigroup $S = \mathbb{N}$, with addition. Let π be a semigroup representation given by

$$\pi(1) = S^*$$

where $S^* : l^2 \to l^2$ is the shift operator $(S^*(x))_j = \begin{cases} 0, & j = 1 \\ x_{j-1}, & j \geq 2 \end{cases}$.

1.4 Question. What are the invariant subspaces of l^2 under the action of π?

Cheaply, we see that $\{0\}$ and l^2 are invariant, but additionally, the fact that S^* always leaves a 0 as the first element of a sequence in l^2 leads us to consider the family of subspaces

$$V_k := \{x \in l^2 : x_j = 0 \forall j < k\}.$$

Each V_k is an invariant subspace, but their nesting is suspicious. To check our suspicions, we note that if $k \geq 2$, then we may denote P_k to represent the orthogonal projection onto V_k and see that

$$\begin{aligned} P_k \pi(1) e_{k-1} &= P_k S^* e_{k-1} \\ &= P_k e_k = e_k \\ &\neq 0 \\ &= \pi(1) 0 \\ &= \pi(1) P_k e_{k-1}. \end{aligned}$$

This shows that the nice properties for involutive semigroups regarding projections onto invariant subspaces truly rely on the presence of the involution.

Finishing the proof from last time

Picking up where we left off, we wanted to show that for any representation $(\pi, \mathcal{H})$ of an involutive semigroup S and a closed subspace $E \subset \mathcal{H}$ the following are equivalent:

1. E is invariant under $\pi(S)$
2. $E^{\perp}$ is invariant under $\pi(S)$
3. $P_E\pi(s) = \pi(s)P_E$, $\forall s \in S$.

Proof. We left off last time having shown $1 \iff 2 \implies 3$. So we now assume that the projection onto E commutes with any realization of the representation. Then if we choose $v \in E$ and $s \in S$ we have

$$\pi(s)v = \pi(s)P_E v = P_E\pi(s)v \in E.$$

Thus, E is invariant under $\pi(S)$. □

1.5 Question. How serious is the non-degeneracy issue?

1.6 Answer. Not that serious, as we will see in the next Theorem.

1.7 Theorem. *If $(\pi, \mathcal{H})$ is a representation of an involutive semigroup S then*

$$\mathcal{H}_0 = \{v \in \mathcal{H} : \forall s \in S, \pi(s)v = 0\}$$

is a closed subspace. Moreover, π is non-degenerate when restricted to $\mathcal{H}_0^{\perp}$ and $\mathcal{H}_0 = (\pi(S)\mathcal{H})^{\perp}$.

Proof. To see that $\mathcal{H}_0$ is closed, note that

$$\begin{aligned}\mathcal{H}_0 =& \{v \in \mathcal{H} : \forall s \in S, \pi(s)v = 0\} \\ =& \bigcap_{s\in S}\{v \in \mathcal{H} : \pi(s)v = 0\} \\ =& \bigcap_{s\in S}\ker(\pi(s))\end{aligned}$$

Which is the intersections of closed spaces. Using the relationship between the orthogonal compliment and adjoint we see that,

$$\begin{aligned}\mathcal{H}_0 =& \bigcap_{s\in S}\ker(\pi(s)) = \bigcap_{s\in S}\left((\pi(s))^*\mathcal{H}\right)^{\perp} \\ =& \bigcap_{s\in S}\left(\pi(s^*)\mathcal{H}\right)^{\perp} = \bigcap_{s\in S}\left(\pi(s)\mathcal{H}\right)^{\perp} \\ =& \Big(\bigcup_{s\in S}\pi(s)\mathcal{H}\Big)^{\perp} = \left(\pi(S)\mathcal{H}\right)^{\perp}.\end{aligned}$$

Hence $\mathcal{H}_0^{\perp} = \overline{\operatorname{span}(\pi(S)\mathcal{H})}$. By the previous lemma, we get that $\mathcal{H}_0^{\perp}$ is invariant by the invariance of $\mathcal{H}_0$, and restricting π to $\mathcal{H}_0^{\perp}$ yields a non-degenerate representation by construction as $\pi(S)\mathcal{H} = \pi(S)(\mathcal{H}_0^{\perp})$ so $\operatorname{span}(\pi(S)(\mathcal{H}_0^{\perp}))$ is dense in $\mathcal{H}_0^{\perp}$. □

The preceding theorem gives an example of how to decompose a Hilbert space into invariant subspaces under a representation. Conversely, we may combine representations via the next theorem.

1.8 Theorem. *Let* $(\pi_j, \mathcal{H}_j)$ *be a family of representations of the same involutive semigroup* S *indexed by the set* J*. If* $\sup_{j\in J} \|\pi_j(s)\| < \infty \forall s \in S$*, then for* $v = (v_j)_{j\in J} \in \bigoplus_{j\in J} \mathcal{H}_j$*,*

$$(\pi(s)v)_j \equiv \pi_j(s)v_j$$

defines a representation of S *on* $\mathcal{H} = \bigoplus_{j\in J} \mathcal{H}_j$*.*

Proof. First, we verify that $\pi(s)$ maps $\mathcal{H}$ to itself. For $v \in \mathcal{H}$ and $s \in S$, we have that

$$\begin{aligned}
\|\pi(s)v\|^2 &= \sum_{j\in J} \|\pi_j(s)v_j\|^2 \\
&\le \sum_{j\in J} \|\pi_j(s)\|^2 \|v_j\|^2 \\
&\le \sum_{j\in J} \sup_{k\in J} \|\pi_k(s)\|^2 \|v_j\|^2 \\
&= \sup_{k\in J} \|\pi_k(s)\|^2 \sum_{j\in J} \|v_j\|^2 < \infty.
\end{aligned}$$

By virtue of each π_j being a homomorphism, so too is π. Moreover, we see that

$$\begin{aligned}
\langle \pi(s)v, w\rangle &= \sum_{j\in J} \langle \pi_j(s)v_j, w_j\rangle \\
&= \sum_{j\in J} \langle v_j, (\pi_j(s))^* w_j\rangle \\
&= \sum_{j\in J} \langle v_j, \pi_j(s^*) w_j\rangle \\
&= \langle v, \pi(s^*)w\rangle
\end{aligned}$$

So $(\pi(s))^* = \pi(s^*)$. □

1.9 Definition. The representation that combines $(\pi_j, \mathcal{H}_j)$ as above is called the *direct sum representation* and is denoted by

$$\pi = \bigoplus_{j\in J} \pi_j$$

1.10 Lemma (Decomposition of non-degenerate representations into cyclic components)**.** *Let* $(\pi, \mathcal{H})$ *be a representation of an involutive semigroup* S *and* $v \in \mathcal{H}$*. Then* π *is cyclic when restricted to* $\overline{\operatorname{span}(\pi(S)v)}$*. Additionally, if* π *is non-degenerate, then* $v \in \overline{\operatorname{span}(\pi(S)v)}$*.*

Proof. Let $\mathcal{H}_1 = \overline{\mathrm{span}(\pi(S)v)}$. Then $\mathcal{H}_1$ is invariant under $\pi(S)$ and so is $\mathcal{H}_1^\perp \equiv \mathcal{H}_2$ by our characterization of invariant subspaces. Hence, we may write $v \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ uniquely as $v = v_1 + v_2$ where $v_1 \in \mathcal{H}_1$ and $v_2 \in \mathcal{H}_2$. So if $s \in S$ we know

$$\mathcal{H}_2 \ni \pi(s)v_2 = \pi(s)(v - v_1) = \pi(s)v - \pi(s)v_1 \in \mathcal{H}_1 \implies \pi(s)v_2 \in \mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}.$$

Thus, $\pi(s)v = \pi(s)v_1$, and v_1 is a cyclic vector for $\mathcal{H}_1$. If the representation is non-degenerate, then $\overline{\mathrm{span}(\pi(S)\mathcal{H})} = \mathcal{H}$. So

$$\{0\} = \big(\pi(S)\mathcal{H}\big)^\perp = \big((\pi(S))^*\mathcal{H}\big)^\perp = \ker(\pi(S)) \ni v_2$$

and $v = v_1 \in \mathcal{H}_1$. ☐