Non-degenerate representations, finite dimensional representations, and representations with intertwining operators

Lecture Notes from September 27, 2022
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Last Time

- Restricting a representation to make it non-degenerate
- Direct sum representations
- How to get cyclic representations

Warm up

Recall the Lemma on how to get cyclic representations from last time:

0.0 Lemma. *Take $v \in \mathcal{H}$ and restrict π to $\mathcal{H}_v := \overline{span\ \pi(S)v}$, then π is cyclic when restricted to $\mathcal{H}_v$, and if π is non-degenerate, we have $v \in \mathcal{H}_v$.*

We now compare the above Lemma with the semigroup representation

$$\pi : \mathbb{N} \to B(l^2),$$

$$\pi(1) = S^*,$$

where S^* is the right shift function. We see that, with this example, we ignore the involution condition $\pi(s^*) = (\pi(s))^*, s \in S$, since

$$(\pi(1))^* = (S^*)^* \neq S^* := \pi(1) = \pi(1^*).$$

Thus, when we choose $v = e_1$, we have $\pi(n)e_1 = e_{1+n}$, so

$$\pi(\mathbb{N})e_1 = \{e_{1+n} : n \in \mathbb{N}\} = \{e_m : m \geq 2\},$$

which implies $\mathcal{H}_{e_1} := \overline{span\ \pi(\mathbb{N})e_1}$ is the orthogonal complement of e_1, $\{e_1\}^\perp$. Since $\mathcal{H}_{e_1}$ contains elements starting with at least one zero, we see that $e_1 \notin \mathcal{H}_{e_1}$, and π restricted to $\mathcal{H}_{e_1}$ has an invariant subspace $\mathcal{H}' : \{x \in l^2 : x_1 = x_2 = 0\}, \mathcal{H}' \neq \{e_1\}^\perp$ (specifically, $\mathcal{H}' \subset \{e_1\}^\perp$), and $\mathcal{H}' \neq \{0\}$, meaning π restricted to $\mathcal{H}_{e_1}$ is not irreducible.

1 Non-degenerate Representations

We begin by restating Zorn's Lemma, to be used as the main proof device for the theorem in this section:

1.1 Lemma. *Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains a maximal element.*

1.2 Theorem. *The representation π of an involutive semigroup S is non-degenerate if and only if π is the direct sum of cyclic representations.*

Proof. Let π be non-degenerate. Since $\mathcal{H} = \overline{span\ \pi(S)\mathcal{H}}$, there is a $0 \neq v \in \mathcal{H}$ for which $\mathcal{H}_v := \overline{span\ \pi(S)v} \neq 0$.

Let $\mathcal{M}$ be the set of all families $\{\mathcal{H}_j\}_{j \in J}$ such that $\mathcal{H}_j \perp \mathcal{H}_k$ if $j \neq k$, and each $\mathcal{H}_j$ is a closed subspace. We observe that, because there is a non-zero $v \in \mathcal{H}$ so that $\mathcal{H}_v$ is nontrivial, by the assumption that π is non-degenerate, $\mathcal{M}$ is not zero, and there is a partial ordering on $\mathcal{M}$: Let $\mathcal{K} := (\mathcal{K}_m), m \in \mathbb{N}$ be a chain in $\mathcal{M}, \mathcal{K}_m := \{\mathcal{H}_j^m\}, j \in J^m$. We say

$$\mathcal{K}_n \leq \mathcal{K}_{n+1}$$

if for any $\mathcal{H}_v^n \in \mathcal{K}_n, \mathcal{H}_v^n \in \mathcal{K}_{n+1}$. Then the upper bound for $\mathcal{K}$ is

$$\mathcal{U} := \bigcup_m \{\mathcal{H}_v^m : \mathcal{H}_v^m \in \mathcal{K}_m\}.$$

To see $\mathcal{U} \in \mathcal{M}$, take any two elements $\mathcal{H}_v^n, \mathcal{H}_i^m \in \mathcal{U}$. Then there must be $\mathcal{K}_n$ and $\mathcal{K}_m$ so that $\mathcal{H}_v^n \in \mathcal{K}_n$ and $\mathcal{H}_i^m \in \mathcal{K}_m$. Without loss of generality, assume $\mathcal{K}_m \leq \mathcal{K}_n$, then $\mathcal{H}_i^m \in \mathcal{K}^n$ and thus $\mathcal{H}_i^m \perp \mathcal{H}_v^n$, so $\mathcal{U}$ is an element in $\mathcal{M}$.

Now, since every chain $\mathcal{K}$ has an upper bound in $\mathcal{M}$, Zorn's lemma gives us a maximal element $\{\mathcal{H}_j\}_{j \in J_{max}}$. Letting $\mathcal{H}_1 = \overline{\sum_{j \in J_{max}} \mathcal{H}_j}$, then $\mathcal{H}_1^\perp$ is an invariant subspace (by Lemma from September 22). If $\mathcal{H}_1 \neq \mathcal{H}$, then since π is non-degenerate, there is $0 \neq v \in \mathcal{H}_1^\perp$ such that $\mathcal{H}' := \overline{span\ \pi(S)v} \neq 0$, and $\{\mathcal{H}'\}$ and $\{\mathcal{H}_j\}_{j \in J_{max}}$ form an orthogonal family of closed subspaces, contradicting the maximality assumption. Therefore, the subspace associated with the maximal element in $\mathcal{M}$, $\mathcal{H}_1$, exhausts the whole space $\mathcal{H}$, i.e. $\mathcal{H}_1 = \mathcal{H}$, and $\mathcal{H}_1^\perp = \{0\}$.

Conversely, if $(\pi, \mathcal{H})$ is a direct sum of cyclic representations $(\pi_j, \mathcal{H}_j)$, then $\sum_{j \in J} \mathcal{H}_j$ is dense in $\mathcal{H}$. Since each representation π_j is cyclic,

$$\mathcal{H}_j \subset \overline{span\ \pi(S)\mathcal{H}_j} \subset \overline{span\ \pi(S)\mathcal{H}}.$$

Thus,

$$\sum_{j \in J} \mathcal{H}_j \subset \overline{span\ \pi(S)\mathcal{H}}.$$

Since $\overline{span\ \pi(S)\mathcal{H}}$ is closed and $\sum_{j \in J} \mathcal{H}_j$ is dense in $\mathcal{H}_1$, $\overline{span\ \pi(S)\mathcal{H}} = \mathcal{H}$, so S is non-degenerate. $\square$

2 Finite Dimensional Representations

2.1 Theorem. *Each finite dimensional representation π of an involutive semigroup S is a direct sum of irreducible representations.*

Proof. If $(\pi, \mathcal{H})$ is irreducible, we have nothing to show.

Otherwise, there is a reducible subspace $\mathcal{H}_1$, $\mathcal{H}_1 \neq 0, \mathcal{H}_1 \neq \mathcal{H}$ which is invariant. We obtain $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}$ where both subspaces are invariant and have dimensions less than dimension of $\mathcal{H}$.

Next, we argue with induction over the dimension of $\mathcal{H}$:

For $\dim \mathcal{H} = 1$, π is irreducible since either $\mathcal{H}_1$ or $\mathcal{H}_1^{\perp}$ has to be $\{0\}$ or $\mathcal{H}$.

If $\dim \mathcal{H} > 1$, splitting $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}$ and applying induction hypothesis to subspaces $\mathcal{H}_1$ and $\mathcal{H}_1^{\perp}$ gives that, each of these subspaces is a dicrect sum of subspaces on which π acts irreducibly. One can also envision a tree of splittings. After each split, we check if each summand is irreducible. If not, we keep splitting and checking irreducibility again. After an finite amount of splittings (since $\mathcal{H}$ is of finite dimension), we must arrive at the irreducible representations. $\square$

3 Representations with intertwining operator

We first recall the definition of intertwining operators:

3.1 Definition. An operator $U \in B(\mathcal{H}, \mathcal{H}')$ is called *intertwining* if, for two representations $(\pi, \mathcal{H}), (\pi', \mathcal{H}')$ of an involutive semigroup S,

$$U \circ \pi(s) = \pi'(s) \circ U$$

for all $s \in S$.

3.2 Lemma. *Given a representation $(\pi, \mathcal{H})$ of a semigroup with involution S, $A \in B(\mathcal{H})$ an intertwining operator, and*

$$\mathcal{H}_\lambda(A) := \{v \in \mathcal{H} : Av = \lambda v\},$$

then $\mathcal{H}_\lambda(A)$ is invariant under S.

Proof. Using the above definition, for $v \in \mathcal{H}_\lambda(A), s \in S$

$$A(\pi(s)v) = \pi(s)Av = \pi(s)\lambda v = \lambda(\pi(s)v),$$

which implies that $\pi(s)v \in \mathcal{H}_\lambda(A)$, so $\mathcal{H}_\lambda(A)$ is invariant under S. $\square$

Next, we consider the case where S is abelian.

3.3 Theorem. *If S is abelian, then each irreducible finite dimensional representation is one dimensional.*

Proof. Consider $s \in S$ and $\pi(s)$. By $\mathcal{H}$ being complex, the characteristic polynomial has at least one root, so there is a $\lambda \in \mathbb{C}$ such that $\mathcal{H}_\lambda(\pi(s)) \neq 0$.

Since S is abelian, $\pi(s)$ intertwines π, and by the above Lemma, $\mathcal{H}_\lambda(\pi(s))$ is invariant under S. By the irreducibility of π, $\mathcal{H}_\lambda(\pi(s)) := \{v \in \mathcal{H} : \pi(s)v = \lambda v\} = \mathcal{H}$, so

$$\pi(s) = \lambda id_{\mathcal{H}}.$$

We conclude that $\pi(S) \subset \mathbb{C}id_{\mathcal{H}}$, but π is irreducible, i.e. $\{0\}$ and $\mathcal{H}_\lambda(\lambda id_{\mathcal{H}})$ are the only closed subspaces that are invariant under $\pi(S)$, so there cannot be an orthogonal projection P so that $0 \neq P \neq id_{\mathcal{H}}$ with $P\pi(s) = \pi(s)P$ (by the Lemma on September 20), and hence $\dim \mathcal{H}$ must be 1, since if such P existed, then $\mathcal{H}$ would be the direct sum of two nontrivial orthogonal subspaces, making $\dim \mathcal{H} \geq 2$. $\square$