Lecture Notes from September 27, 2022

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Last time

- Restricting a representation to make it non-degenerate
- Direct sum representations
- How to get cyclic representations

Recall the following:

1.6 Lemma. *Take $v \in \mathcal{H}$, restrict π to $\mathcal{H}_v \cong \overline{\text{span}\ \pi(S)v}$. The π is cyclic when restricted to $\mathcal{H}_v$. If π is non-degenerate then $v \in \mathcal{H}_v$.*

Warm up:

1.7 Question. What happens if S is not involutive?

Compare the lemma with the following example: $\pi : \mathbb{N} \to B(l^2)$. Where $\pi(1) = S^*$ is the right shift.

Let $v = e_1$. Then, $\pi(n)v = e_{1+n}$. Thus, $\overline{\text{span}\pi(S)v} = \{(0, x_2, x_3, ...)\} = \{e_1\}^\perp = \mathcal{H}_{e_1}$, $e_1 \notin \{e_1\}^\perp$. Also, $\pi|_{\mathcal{H}_{e_1}}$ has infinitely many invariant subspaces. One of them is $\mathcal{H}^2 = \{x \in l^2 : x_1 = x_2 = 0\}$. $\mathcal{H}^2 \neq \{e_1\}^\perp$, and $\mathcal{H}^2 \neq \{0\}$. Thus, $\pi|_{\mathcal{H}_{e_1}}$ is not irreducible. Also, $\pi|_{\mathcal{H}_{e_1}}$ can be seen to not be cyclic. If we take $w \in \mathcal{H}_{e_1}$ then for all $s \in S$, $\pi(s)w \subset \mathcal{H}^2$. In particular, $e_2 \notin \overline{\text{span}\pi(S)w}$.

We now turn back to the case of representations of involutive semigroups and the linear structure of nondegenerate representations.

1.8 Theorem. *The representation of an involutive semigroup, S, is nondegenerate iff it is the direct sum of cyclic representations.*

Proof. Let the representation be non-degenerate. If it is not, we can "make" it non-degenerate by the first theorem of September 22. That is, we can remove, in the sense of direct sums, the intersection of the kernels of $\pi(s)$ for each $s \in S$.

Since our representation is nondegenerate, $\mathcal{H} = \overline{\text{span}\ \pi(S)\mathcal{H}}$, so there exists a vector $v \in \mathcal{H}$ for which $\overline{\text{span}\ \pi(S)v}$ is not zero. We are looking to apply Zorn's Lemma as a countable process need not "exhaust" the Hilbert Space $\mathcal{H}$. Let $\mathcal{M}$ be the set of all indexed families of closed mutually orthogonal cyclic subspaces of $\mathcal{H}$ that are invariant under π. That is, $\mathcal{M}$ is the set of $\{\mathcal{H}_j\}_{j \in \mathcal{J}}$ such that $\mathcal{H}_j \perp \mathcal{H}_k$ for $k \neq j$, each $\mathcal{H}_j$ is a closed subspace, is cyclic, and is invariant under π. We can define a partial order on $\mathcal{M}$ by $A_1 < A_2$ if $A_1 \subset A_2$. Here we are saying that if $\mathcal{H}_j \in A_1$ then $\mathcal{H}_j \in A_2$. We note that for each chain, $\mathcal{K} = (K_m)$, $K = \bigcup_m \{\mathcal{H}_j : \mathcal{H}_j \in K_m\}$ is an upper bound. This follows from its construction. If $\mathcal{H}_j \in K_m$ for any $m \in \mathbb{N}$, then $\mathcal{H}_j \in K$. Hence, $K_m < K$ for all $m \in \mathbb{N}$. If $\mathcal{H}_j$ and $\mathcal{H}_k$ are in K then there is an $m \in \mathbb{N}$ that contain them both. Thus, if $\mathcal{H}_j \neq \mathcal{H}_k$, $\mathcal{H}_j \perp \mathcal{H}_k$. Also, for the same reason, these $\mathcal{H}_j$ are cyclic and invariant under π, which shows K is indeed an element of $\mathcal{M}$. Thus, by Zorn's Lemma, there exists a maximal element $\mathcal{A} \in \mathcal{M}$. By maximal we mean, if $\mathcal{A} < B$ for some $B \in \mathcal{M}$ then $B = \mathcal{A}$. Denote $\mathcal{A} = \{\mathcal{H}_j\}_{j \in \mathcal{J}_{max}}$. Let $\mathcal{H}_1 = \overline{\sum_{j \in \mathcal{J}_{max}} \mathcal{H}_j}$. Each $\mathcal{H}_j$ is an invariant subspace, so $\mathcal{H}_1$ is invariant. This took me ten minutes of thinking before I concluded it was obvious. If you are like me then let me spare you some of those minutes. If $w \in \mathcal{H}_1$ then there exists a sequence w_i in $\sum_{j \in \mathcal{J}_{max}} \mathcal{H}_j$ converging to w. Thus, for any $s \in S$, $\pi(s)w_i \in \sum_{j \in \mathcal{J}_{max}} \mathcal{H}_j$ for all i by the invariance of each $\mathcal{H}_j$ and the linearity of $\pi(s)$. By the continuity of $\pi(s)$, $\pi(s)w_i$ converges to $\pi(s)w$. Hence, $\pi(s)w \in \overline{\sum_{j \in \mathcal{J}_{max}} \mathcal{H}_j}$. Then, $\mathcal{H}_1^\perp$ is an invariant subspace. If $\mathcal{H}_1 \neq \mathcal{H}$, then there exists a nonzero $v \in \mathcal{H}_1^\perp$. Let $\mathcal{H}_c = \overline{\text{span }\pi(S)v}$. This closed subspace is invariant and cyclic by construction. Thus, $\{\mathcal{H}_c\} \bigcup \{\mathcal{H}_j\}_{j \in \mathcal{J}_{max}}$ is an orthogonal family of closed, invariant, cyclic subspaces that properly contains $\mathcal{A}$. This contradicts Zorn's lemma. Therefore, $\mathcal{H}_1 = \mathcal{H}$.

Conversely, if $(\pi, \mathcal{H})$, is a direct sum of cyclic representations, $(\pi_j, \mathcal{H}_j)$, then $\sum_{j \in \mathcal{J}} \mathcal{H}_j$ is dense in $\mathcal{H}$. The representation of each π_j is cyclic, so

$$\mathcal{H}_j \subset \overline{\text{span }\pi(S)\mathcal{H}_j} \subset \overline{\text{span }\pi(S)\mathcal{H}}.$$

Hence, if we sum over j and take the closure, we obtain

$$\mathcal{H} = \overline{\sum_{j \in \mathcal{J}} \mathcal{H}_j} \subset \overline{\sum_{j \in \mathcal{J}} \overline{\text{span }\pi(S)\mathcal{H}_j}} \subset \overline{\sum_{j \in \mathcal{J}} \overline{\text{span }\pi(S)\mathcal{H}}} = \overline{\text{span }\pi(S)\mathcal{H}} \subset \mathcal{H}.$$

Relying on the the maxim $\mathcal{H} = \mathcal{H}$, we have shown $\mathcal{H} = \overline{\text{span }\pi(S)\mathcal{H}}$, so $\mathcal{H}$ is nondegenerate. □

Having done the 'hard' work of unrestricted dimensions, we now turn to the finite dimensional case. In this setting we will arrive at direct sums of irreducible representations, which is a stronger condition than cyclic.

1.9 Remark. A nontrivial irreducible representation of an involutive semigroup is cyclic, but a cyclic representation need not be irreducible.

Proof. Suppose we have a nontrivial irreducible representation $(\pi, \mathcal{H})$ of an involutive semigroup S. That is, the only invariant subspaces of π are $\{0\}$ and $\mathcal{H}$ and there exists an $s \in S$ and a $v \in \mathcal{H}$ such that $\pi(s)v \neq 0$. Then, $W = \overline{\text{span }\pi(S)v} = \mathcal{H}$. This follows from the fact that W is a nonzero closed subspace of $\mathcal{H}$ that is invariant under π. However, given $A = [e3, e1, e2]$, the set $\{I, A, A^2\}$ is a cyclic representation of $\mathbb{Z}_3$ in $\mathcal{H} = \mathbb{R}^3$. It is cyclic under e_1 because for any $w \in \mathcal{H}$, $w = w_1 I e_1 + w_2 A^2 e_1 + w_3 A e_1$. It is not irreducible because the line in $\mathbb{R}^3$ defined by $\{av : a \in \mathbb{R}, v = (1,1,1)^t\}$ is invariant under π because $Av = v$. □

1.10 Theorem. *Each finite dimensional representation of an involutive semigroup, S, is a direct sum of irreducible representations.*

Proof. If $(\pi, \mathcal{H})$ is irreducible, there is nothing to prove.

Suppose $(\pi, \mathcal{H})$ is not irreducible. Then, there exists an invariant subspace $\mathcal{H}_1$, such that $\mathcal{H}_1 \neq \{0\}$ and $\mathcal{H}_1 \neq \mathcal{H}$. We also have that $\mathcal{H}_1^\perp$ is invariant by the Lemma from September 20, and $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_1^\perp$.

We are now ready to argue with induction over $\dim \mathcal{H}$. If $\dim \mathcal{H} = 1$ then π is irreducible. If $\dim \mathcal{H} > 1$ we can split the Hilbert space $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_1^\perp$ into two subspaces each with dimension at least 1 and at most $\dim \mathcal{H} - 1$. We now ask if $\mathcal{H}_1$ and $\mathcal{H}_1^\perp$ are irreducible. If either of them are not, we can again split, which will again reduce the dimension by at least 1. Since $\dim \mathcal{H}$ is finite, this process will eventually terminate with a direct sum of ~~prime~~ irreducible subspaces. Restricting π to each of these irreducible subspaces forms the direct sum of irreducible representations. $\square$

We continue towards a description of the structure of the irreducible representations. First, a lemma.

1.11 Lemma. *Given a representation of an involutive semigroup, S, and an intertwining operator $A \in \mathrm{B}(\mathcal{H})$ an intertwining operator, the closed subspace $\mathcal{H}_\lambda(A) = \{v \in \mathcal{H} : Av = \lambda v\}$ is invariant under S.*

Proof. For $v \in \mathcal{H}_\lambda(A)$, and $s \in S$,

$$A\pi(s)v = \pi(s)Av = \lambda\pi(s)v.$$

Hence, $\pi(s)v \in \mathcal{H}_\lambda(A)$, so $\mathcal{H}_\lambda(A)$ is invariant under S. $\square$

We are now prepared to describe the irreducible representations of abelian involutive semigroups.

1.12 Theorem. *If S is abelian, then each irreducible finite representation is 1-dimensional.*

Proof. Let $s \in S$. Then, the operator $\pi(s)$ has a characteristic polynomial, which, since $\mathcal{H}$ is complex, has at least one root. Thus, there exists a $\lambda \in \mathbb{C}$ such that $\mathcal{H}_\lambda(\pi(s)) \neq \{0\}$. Since S is abelian, $\pi(s)$ intertwines and hence $\mathcal{H}_\lambda(\pi(s))$ is invariant under π. By the irreducibility of π, $\mathcal{H}_\lambda(\pi(s)) = \mathcal{H}$ so $\pi(s) = \lambda \mathrm{Id}_{\mathcal{H}}$. We conclude $\pi(S) \subset \mathbb{C}\mathrm{Id}_{\mathcal{H}}$.

If the dimension of $\mathcal{H}$ is greater than one, then we can take a $v \neq 0 \in \mathcal{H}$ and form a one dimensional subspace of $\mathcal{H}$ by $\mathrm{span}(v) = V$. However, for any $s \in S$, $\pi(s)v = \lambda v$, so V is invariant under π. But V is one dimensional and $\mathcal{H}$ is not, which contradicts the irreducibility of π. Therefore, the dimension of $\mathcal{H}$ is one. $\square$