Lecture Notes from September 29, 2022

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Last Time

- π is non-degenerate iff π is the direct sum of cyclic representations.
- Irreducible finite-dimensional representations of abelian semigroups.

Warm up:

1.0 Question. Consider the semigroup $S = \{0, 1\}$ with multiplication. Let $s^* = s$. Given a representation $(\pi, \mathcal{H})$ with $\dim \mathcal{H} = n$, how many different representations are there up to unitary equivalence?

We know $\pi(1) = \mathcal{P}$, an orthogonal projection, by $\pi(1)(\pi(1)^*) = \pi(11^*) = \pi(1)$. If π is non-degenerate, we claim that $\mathcal{P}$ is onto, i.e. $\mathcal{P} = \mathrm{id}_{\mathcal{H}}$. Since $\mathcal{H}$ is finite-dimensional, the claim follows from

$$\overline{\pi(S)(\mathcal{H})} = \pi(S)(\mathcal{H}) = \pi(1)\pi(S)(\mathcal{H}) = \mathcal{P}\pi(S)(\mathcal{H}) = \mathcal{P}(\mathcal{H}).$$

And from $0 = 00^*$, we also know that $\pi(0)$ is an orthogonal projection.

In the degenerate case, we have $\mathcal{P} = \pi(1)$ is an orthogonal projection onto some subspace of $\mathcal{H}$, and $\mathcal{Q} = \pi(0)$ is an orthogonal projection onto a subspace of $\pi(1)(\mathcal{H})$.

Up to unitary equivalence, there is only one orthogonal projection onto a subspace of dimension k for each $k \leq n$. So for each projection $\mathcal{P}$ of dimension k, there are $k + 1$ choices of $\mathcal{Q}$.

1.1 Answer. Hence, the number of distinct representations up to unitary equivalence is given by the following sum:

$$\sum_{k=0}^{n} (k+1) = \frac{(n+1)(n+2)}{2}.$$

Characters of Semigroups

1.2 Definition. A representation $\pi : S \to \mathcal{B}(\mathbb{C}) \cong \mathbb{C}$ is called a *character* of S, and we write $\hat{S}_0$ for all such characters, and $\hat{S} \equiv \hat{S}_0 \setminus \{0\}$.

We recall the following two results from the lecture on September 29.

1.3 Theorem. *Each finite dimensional representation π of an involutive semigroup S is a direct sum of irreducible representations.*

1.4 Theorem. *If S is abelian, then each irreducible finite-dimensional representation is one-dimensional.*

Now we can prove the following decomposition theorem.

1.5 Theorem. *Let $(\pi, \mathcal{H})$ be a finite-dimensional representation of an abelian involutive semigroup S. For $\chi \in \hat{S}$, we let*

$$\mathcal{H}_\chi = \{v \in \mathcal{H} : (\forall s \in S)\ \pi(s)v = \chi(s)v\},$$

then $\mathcal{H} = \bigoplus_{\chi \in \hat{S}} \mathcal{H}_\chi$. (Note that at most finitely many $\mathcal{H}_\chi \neq \{0\}$.)

Proof. From Theorems 1.3 and 1.4, we know that each finite-dimensional representation is a direct sum of irreducible representations, and we also know that each irreducible representation of an abelian semigroup is one-dimensional.

We employ some arguments from the proof of Theorem 1.4.

Consider one of these one-dimensional representations $(\pi_j, \mathcal{H}_j)$. Then for any $\pi_j(s)$, there is a $\lambda_s \in \mathbb{C}$ and $v \in \mathcal{H}_j$ (spanning this space) such that $\pi_j(s)v = \lambda_s v$. Define the character $\chi_j : S \to \mathcal{B}(\mathbb{C})$ by $\chi_j(s) = \lambda_s \mathrm{id}_{\mathcal{H}_j}$. Then $\pi_j(s)v = \chi_j(s)v$ for all $s \in S$. And since v spans $\mathcal{H}_j$, $\pi_j(s)v = \chi_j(s)v$ for all $v \in \mathcal{H}_j$. Therefore, $\mathcal{H}_j = \mathcal{H}_{\chi_j}$.

Hence, $\mathcal{H} = \bigoplus_{j \in J} \mathcal{H}_j = \bigoplus_{j \in J} \mathcal{H}_{\chi_j} = \bigoplus_{\chi \in \hat{S}} \mathcal{H}_\chi$. □

We conclude with a classification result, which

1.6 Theorem. *Let S be an abelian involutive semigroup, then each finite-dimensional representation $(\pi, \mathcal{H})$ has a multiplicity function*

$$n_\pi : \hat{S} \to \mathbb{N} \cup \{0\}$$

$$\chi \mapsto \dim \mathcal{H}_\chi$$

and

1. *two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are equivalent iff n_π and n'_π are identical,*
2. *if $n : \hat{S} \to \mathbb{N} \cup \{0\}$ is non-vanishing for finitely many χ, then there is $(\pi, \mathcal{H})$ with $n_\pi = n$.*

Proof.

1. $(\Rightarrow)$ If $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are equivalent, then there is a unitary $\mathcal{U} : \mathcal{H} \to \mathcal{H}'$, $\mathcal{U}(\mathcal{H}) = \mathcal{H}'$, and for any character $\chi \in \hat{S}$, $\mathcal{U}(\mathcal{H}_\chi) = \mathcal{H}'_\chi$, because if $v \in \mathcal{H}_\chi$, $s \in S$,

$$\pi'(s)\mathcal{U}v = \mathcal{U}\pi(s)v = \mathcal{U}\chi(s)v = \chi(s)\mathcal{U}v,$$

so $\mathcal{U}v \in \mathcal{H}'_\chi$.

Conversely, given $v' \in \mathcal{H}'_\chi$, then

$$\pi(s)\mathcal{U}^*v' = \mathcal{U}^*\pi'(s)v' = \mathcal{U}^*\chi(s)v' = \chi(s)\mathcal{U}^*v'.$$

Hence $\mathcal{U}^*v' \in \mathcal{H}_\chi$. Since $\mathcal{U}^* = \mathcal{U}^{-1}$, this establishes $\mathcal{U}(\mathcal{H}_\chi) = \mathcal{H}'_\chi$ and consequently, $n_\pi(\chi) = \dim \mathcal{H}_\chi = \dim \mathcal{H}'_\chi = n'_\pi(\chi)$.

($\Leftarrow$) Conversely, given two representations and for each $\chi \in \hat{S}$, $\mathcal{H}_\chi$ and $\mathcal{H}'_\chi$ have the same dimension, then they are isomorphic. Consider the isomorphism $\mathcal{U}_\chi : \mathcal{H}_\chi \to \mathcal{H}'_\chi$. Then $\mathcal{U}_\chi$ intertwines $\pi\mid_{\mathcal{H}_\chi}$ and $\pi\mid_{\mathcal{H}'_\chi}$.

Next, let $\mathcal{U} : \mathcal{H} \to \mathcal{H}'$, $\mathcal{U}\mid_{\mathcal{H}_\chi} = \mathcal{U}_\chi$, then $\mathcal{U}$ intertwines π and π' on the direct sum spaces $\mathcal{H}$ and $\mathcal{H}'$. Thus, π and π' are equivalent.

2. Given n as described, let $\mathcal{H}_\chi = \mathbb{C}^{n(\chi)}$ and define π_χ on $\mathcal{H}_\chi$ by

$$\pi_\chi(s) = \chi(s)\mathrm{id}_{\mathcal{H}_\chi}.$$

Since n is only nonzero for finitely many χ, $\bigoplus_{\chi \in \hat{S}} \pi_\chi$ defines a finite-dimensional representation with multiplicity function n.

□

1.7 Remark. The above theorem extends the statement "All Hilbert spaces of the same dimension are unitarily equivalent" to equivalence of representations when the dimension of subspaces $\mathcal{H}_\chi$ are equivalent.