Lecture Notes from October 4, 2022

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Last time

- Characterization of representations of abelian semigroups

Warm up:
Given $G = \mathbb{Z}$ (additive) and $s^* = -s$, show

$$\hat{G} \cong S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

To see this, suppose there is a homomorphism $\chi : G \mapsto \mathbb{C}$ and it is not identically zero. Then we know $\chi(0)$ is an orthonormal projection so $\chi(0) \in \{0, 1\}$ but then by χ not identically zero and

$$\chi(n) = \chi(n + 0) = \chi(0)\chi(n)$$

we have $\chi(0) = 1$ and otherwise trivial.
We also know

$$\chi(n) = \begin{cases} (\chi(1))^n & \text{if } n \in \mathbb{N}_0 \\ (\chi(1)^*)^n & \text{if } n < 0 \end{cases}$$

Now let $z = \chi(1)$ then

$$(\chi(1))^*(\chi(1)) = (\chi(-1))(\chi(1)) = \chi(-1 + 1) = \chi(0) = 1$$

This implies that $\bar{z}z = 1$, so $|z| = 1$
Hence, every χ is characterized by $\chi(1) = z \in S^1$
Conversely given any $z \in S$, assigning $\chi(1) = z$ yields a character on S.
Moreover, if $S = (\mathbb{N}_0, +)$, and $s = s^*$ then $\hat{S}_0 = \mathbb{R}$.
And if $S = (\mathbb{N}_0 \times \mathbb{N}_0, +)$, with $(n, m)^* = (m, n)^*$ then $\hat{S}_0 = \mathbb{C}$

We'll conclude the warm up with a spectral theorem for normal operators on finite dimensional Hilbert Spaces.

1.1 Theorem. *Let $\dim \mathcal{H} < \infty$. An operator $A \in \mathcal{B}(\mathcal{H})$ is normal if and only if*

$$\mathcal{H} = \bigoplus_{\lambda} \mathcal{H}_\lambda$$

where λ denumerates the eigenvalues of A.

Proof. If A is normal then we define an involutive semigroup representation for $S = \{(n, m) : n, m \in \mathbb{N}_0\}$ with $(n, m)^* = (m, n)$ by $\pi(n, m) = A^n(A^*)^m$

Since S is abelian, the representation decomposes into a direct sum of one-dimensional ones on invariant subspaces that are mutually orthogonal.

Hence, A is diagonalizable and the eigenspaces of A are the invariant subspaces.

Conversely, if $\mathcal{H}$ is a direct sum of eigenspaces for A, taking $v \in \mathcal{H}_\lambda$ so $Av = \lambda v$ then gives $A|_{span\{v\}} = \lambda id_{span\{v\}}$ so $A^*|_{span\{v\}} = \bar{\lambda} id_{span\{v\}}$

So on each eigenspace, the restriction of A and A^* commute so by the direct sum decomposition, $AA^* = A^*A$ and hence A is normal.

□

1.2 Definition. A complex vector space A with a map $A \times A \mapsto A, (x, y) \mapsto xy$ is called an (associative) algebra if $(xy)z = x(yz)$ for each $x, y, z \in A$

An element 1 is called a unit if $1a = a1 = a$ for each $a \in A$

If A has a unit, then an element $a \in A$ is called invertible if there is $b \in A$ such that $ab = ba = 1$. We can show that the inverse b is unique by supposing it is not unique and showing this leads to a contradiction.

Let there be $a, b, c \in A$ such b, c are each the inverse of a and $b \neq c$ then we have $ab = ba = 1$ and $ac = ca = 1$. This gives us that

$$\begin{aligned} ab &= ac \\ ab - ac &= 0 \\ a(b - c) &= 0, \quad \forall a \in A \\ b - c &= 0 \\ b &= c \end{aligned}$$

and thus we have a contradiction, and therefore b is unique. We then say b is the inverse of a and $b = a^{-1}$

The set $G(A)$ of invertible elements forms a group with unit 1.

An algebra A which is a Banach space is called a Banach algebra is $||ab|| \leq ||a||||b||$ for $a, b \in A$

1.3 Lemma. *Multiplication in a Banach algebra is continuous.*

Proof. Let $a_n \to a, b_n \to b$,

$$\begin{aligned} ||a_n b_n - ab|| &= ||a_n b_n - ab_n + ab_n - ab|| \\ &\leq ||a_n b_n - ab_n|| + ||ab_n - ab|| \\ &\leq \underbrace{||a_n - a||}_{\to 0} \underbrace{||b_n||}_{\text{stays bounded}} + ||a|| \underbrace{||b_n - b||}_{\to 0} \end{aligned}$$

and by $(||b_n||)_{n=1}^\infty$ being bounded, we get $||a_n b_n - ab|| \to 0$ □

1.4 Definition. 1. An involutive algebra A is an (associative) complex algebra for which there is a representation $a \mapsto a^*$ such that for each $a, b \in A$, and $\lambda, \mu \in \mathbb{C}$

(a) $(a^*)^* = a$

(b) $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$

(c) $(ab)^* = b^* a^*$

2. If $(A, \|\cdot\|)$ is a Banach algebra with involution and $\|a^*\| = \|a\|$ for each $a \in A$, then we say that A is a Banach-$*$-algebra.

   If it is even true that for each $a \in A$, $\|aa^*\| = \|a\|^2$, then this is called a C^*-algebra.

3. If $(A, *)$ is an involutive algebra, then $\hat{A}$ is the set of non-zero homomorphisms of A to $\mathbb{C}$. If A is a Banach-$*$-algebra, then we write $\hat{A}$ for the continuous non-zero homomorphisms.

4. An element $a \in A$, with A an involutive algebra, is called

   (a) normal if $aa^* = a^* a$

   (b) Hermitian if $a = a^*$

   (c) orthogonal projection if $aa^* = a$

1.5 Remark. If $\mathcal{H}$ is a complex Hilbert space, then closed subset $A \subset \mathcal{B}(\mathcal{H})$ forms an algebra with adjoint as an involution $a \mapsto a^*$

(i.e. $A^* \subset A$) then A is a C^*-algebra.

In particular, $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. This is the case because on $\mathcal{B}(\mathcal{H})$, we had shown $\|a^* a\| = \|a\|^2 = \|a^*\|^2$, for each $a \in \mathcal{B}(\mathcal{H})$