Lecture Notes from October 10, 2022

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Last time

- characterization of representations of abelian semigroups

Warm up:

1.6 Question. Given $G = (\mathbb{Z}, +)$ and $s^* = -s$, then show

$$\hat{G} \cong S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

To see this, assume $\chi : G \mapsto \mathbb{C}$ is a character. Then, we know that

$$\chi(0) = [\chi(0)]^* \chi(0),$$

so $\chi(0) \in \{0, 1\}$. Moreover, $\chi(n) = \chi(n+0) = \chi(0)\chi(n)$ for each $n \in \mathbb{Z}$, and χ is not identically zero by definition of a character. Hence, $\chi(0) = 1$. Furthermore,

$$\chi(n) = \begin{cases} [\chi(1)]^n & n \in \mathbb{N}_0 \\ [\chi(1)^*]^n & n < 0 \end{cases}$$

and $(\chi(1))^* \chi(1) = \chi(0)$. Therefore, if we let $z = \chi(1)$, we see that $\bar{z}z = 1$, so $|z| = 1$. Hence, every χ is characterized by $\chi(1) = z \in S^1$. Conversely, given any $z \in S^1$ assigning $\chi(1) = z$ yields a character on G.

1.7 Question. Given $S = (\mathbb{N}_0, +)$ and $s^* = s$, then show that $\hat{S}_0 \cong \mathbb{R}$

Note, if $\chi \in \hat{S}_0$ is the zero homomorphism then we can identify χ with the real number zero. So, assume $\chi : S \mapsto \mathbb{C}$ is a character. Using similar techniques as in the previous question, we can show that $\chi(0) = 1$ and $\chi(n) = [\chi(1)]^n$ for each $n \in \mathbb{N}_0$. Furthermore,

$$\begin{aligned} \chi(n) &= \chi(n^*) \\ &= \chi(n)^* \end{aligned}$$

for each $n \in \mathbb{N}_0$. Therefore, if we let $z = \chi(1)$, then $z = \bar{z} \in \mathbb{R}$. Conversely, given any $a \in \mathbb{R}$ assigning $\chi(n) = a^n$ for each $n \in \mathbb{N}_0$ yields a homomorphism in $\hat{S}_0$.

1.8 Question. Given $S = (\mathbb{N}_0 \times \mathbb{N}_0, +)$ and $(n, m)^* = (m, n)$, then show that $\hat{S}_0 \cong \mathbb{C}$

Assume $\chi : S \mapsto \mathbb{C}$ is a character. Since χ is not identically zero, we can show that $\chi((0,0)) = 1$. It now follows from the lecture notes on September 20, 2022 that $\chi((1,0))$ determines the representation/character. Therefore, every χ is characterized by $\chi((1,0)) = z \in \mathbb{C}/\{0\}$. As in the previous question, we identify the zero homomorphism with zero. Conversely, we can show that if we are given $z \in \mathbb{C}$ that setting $\chi((m,n)) = z^m \overline{z}^n$ for each $m, n \in \mathbb{N}_0$ determines a homomorphism in $\hat{S}_0$.

We conclude with a spectral theorem for normal operators on finite dimensional Hilbert spaces.

1.9 Theorem. *Let $\mathcal{H}$ be a complex Hilbert space such that $\dim \mathcal{H} < \infty$. An operator $A \in B(\mathcal{H})$ is normal if and only if $\mathcal{H} = \bigoplus_{\lambda} \mathcal{H}_\lambda$ where λ denumerates the eigenvalues of A*

Proof. If A is normal, then we define an involutive semigroup representation for

$$S = \{(n,m) : n, m \in \mathbb{N}_0\}$$

with $(n,m)^* = (m,n)$ by

$$\pi(n,m) = A^n (A^*)^m.$$

Since S is abelian, the representation decomposes into a direct sum of one-dimensional ones on invariant subspaces that are mutually orthogonal. Hence, A is diagonalizable and the eigenspaces of A are invariant subspaces.

Conversely, if $\mathcal{H}$ is a direct sum of eigenspaces for A, taking $v \in \mathcal{H}_\lambda$, i.e. $Av = \lambda v$, then by $A|_{span\{v\}} = \lambda id_{span\{v\}}$, we have $A^*|_{span\{v\}} = \overline{\lambda} id_{span\{v\}}$. On each eigenspace, the restriction of A and A^* commute, so by direct sum decomposition, $AA^* = A^*A$. Hence, A is normal. □

2 Banach algebras and Spectral Theory

2.1 Definition.

1. A complex vector space A with a map $A \times A \mapsto A$, $(x,y) \mapsto xy$ is called an *(associative) algebra* if $(xy)z = x(yz)$ for each $x, y, z \in A$.
2. Let A be an algebra. An element 1 is called a *unit* if $1a = a1 = a$ for each $a \in A$.
3. Let A be an associative algebra with a unit. An element $a \in A$ is called *invertible* if there is $b \in A$ such that $ab = ba = 1$. In that case, the inverse is unique. We then say b is the inverse of a, $b = a^{-1}$.
4. An algebra A which is a Banach space is called a *Banach algebra* if $\|ab\| \leq \|a\|\|b\|$ for $a, b \in A$.

2.2 Remark. The set $G(A)$ of invertible elements forms a group with unit 1.

2.3 Claim. *Let A be an associative algebra with a unit. If $a \in A$ is invertible then the inverse is unique.*

Proof. Suppose a is invertible and that b_1 and b_2 are inverses of A. Observe that,

$$\begin{aligned} b_1 &= b_1 1 \\ &= b_1(ab_2) \\ &= (b_1 a)b_2 \\ &= 1b_2 \\ &= b_2 \end{aligned}$$

This shows the inverse is unique. □

2.4 Lemma. *Multiplication in a Banach algebra is continous.*

Proof. Let $a_n \to a$ and $b_n \to b$. Using the definition of a Banach algebra and the triangle inequality, we have

$$\begin{aligned} \|a_n b_n - ab\| &= \|a_n b_n - ab_n + ab_n - ab\| \\ &\leq \|a_n b_n - ab_n\| + \|ab_n - ab\| \\ &\leq \|a_n - a\|\|b_n\| + \|a\|\|b_n - b\| \end{aligned}$$

Note $(\|b_n\|)_{n=1}^{\infty}$ is bounded due to the assumption that $b_n \to b$. Using this and that the two sequences converge by assumption, it follows that $\|a_n b_n - ab\| \to 0$. □

2.5 Definition. 1. An *involutive algebra* A is a (associative) complex algebra for which there is a map $a \mapsto a^*$ such that for each $a, b \in A$, $\lambda, \mu \in \mathbb{C}$,

   (a) $(a^*)^* = a$,

   (b) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$

   (c) $(ab)^* = b^* a^*$

2. If $(A, \|\cdot\|)$ is a Banach algebra with involution and $\|a^*\| = \|a\|$ for each $a \in A$, then we say that A is a *Banach-*-algebra*.

3. If $(A, \|\cdot\|)$ is a Banach-*-algebra with the property that $\|aa^*\| = \|a\|^2$ for each $a \in A$, then this is called a C^**-algebra*.

4. If $(A, *)$ is an involutive algebra, then $\hat{A}$ is the set of non-zero homomorphism of A to $\mathbb{C}$.

5. If A is a Banach-*-algebra, then we write $\hat{A}$ for the set containing the continous non-zero homomorphisms.

6. An element $a \in A$, A an involutive algebra, is called

   (a) normal if $aa^* = a^*a$,

   (b) Hermitian if $a = a^*$,

   (c) orthogonal projection if $aa^* = a$.

2.6 Remark. If $\mathcal{H}$ is a complex Hilbert space and $A \subset B(\mathcal{H})$ is a closed subset which forms an algebra with the adjoint as involution, $a \mapsto a^*$, i.e. $A^* \subset A$, then A is a C^*-algebra. In particular, $B(\mathcal{H})$ is a C^*-algebra. This is the case because on $B(\mathcal{H})$, we had shown $\|a^*a\| = \|a\|^2 = \|a^*\|^2$ for each $a \in B(\mathcal{H})$.