Lecture Notes from October 06, 2022

taken by Phuong Tran

Last time

- The spectral theorem for normal operator on finite dimensional Hilbert spaces,
- Involutive algebras, Banach-*-algebra , C^*-algebra

Warm up:

2.2 Question. If A is involutive algebra, and $\mathbb{1}$ is a left unit i.e. $\mathbb{1}a = a$ for each $a \in A$, then show $\mathbb{1}$ is unique left unit and it is also right unit.

- right unit: we start from $\mathbb{1}\mathbb{1}^* = \mathbb{1}^*$, taking $(.)^*$ both sides, we obtain

$$(\mathbb{1}\mathbb{1}^*)^* = (\mathbb{1}^*)^* \implies \mathbb{1}\mathbb{1}^* = \mathbb{1}$$

  Thus $\mathbb{1} = \mathbb{1}^*$. We then have $a\,\mathbb{1} = (\mathbb{1}^* a^*)^* = (a^*)^* = a$.

- uniqueness: Now we see if there is (another) left unit **e**, then

$$\mathbf{e} \overset{\mathbb{1} \text{ is right unit}}{=} \mathbf{e}\mathbb{1} \overset{\text{e is left unit}}{=} \mathbb{1}$$

2.3 Lemma. *Let A be an involutive algebra, then the following properties hold:*

(1) Hermitian elements in A are normal,

(2) An element of the form xx^* for $x \in A$, is Hermitian

(3) The product of two Hermitian elements x and y is Hermitian if and only if $xy = yx$

(4) $A = A_n \oplus iA_n$ i.e. each $a \in A$ has a unique decomposition $a = b + ic$ with b, c Hermitian

(5) An element $a = b + c,\ b = b^*,\ c = c^*$ is normal if and only if $bc = cb$

(6) A has a unit, and x has an inverse, then $(x^{-1})^* = (x^*)^{-1}$

(7) If $\|.\|$ is a sub-multiplicative norm on A and $\|x\|^2 \le \|x^* x\|$ for each $x \in A$, then $\|x^*\| = \|x\|$ and $\|x^* x\| = \|x\|^2$

2.4 Remark. iA_n is all the skew Hermitian with A_n is Hermitian

Proof.

(1) Let $a \in A$ be Hermitian i.e $a = a^*$, then $a\,a^* = a\,a = a^*\,a^* = a^*\,a$. So a is normal.

(2) Let $x \in A$. Then we have $(x\,x^*)^* = (x^*)^*\,x^* = x\,x^*$, so $x\,x^*$ is Hermitian.

(3) Let $x, y \in A$ be Hermitian.
If $xy = yx$, then $(xy)^* = (yx)^* \implies (xy)^* = x^*y^* = xy$ since $x = x^*$, $y = y^*$.
Conversely, if xy is Hermitian, then $xy = (xy)^* = y^*x^* = yx$ since $x = x^*$, $y = y^*$.

(4) Given $a \in A$, we write $b = \frac{a+a^*}{2}$ and $c = \frac{a-a^*}{2i}$. Then b, c are Hermitian and $a = b + ic$. Moreover, if $a = b' + ic'$ with $(b')^* = b'$ and $(c')^* = c'$ then by taking Hermitian and antiHerimitian parts give $b' = b$ and $c' = c$.

(5) We have

$$\begin{aligned} a\,a^* &= (b+ic)\underbrace{(b-ic)}_{b^*-ic^*} \\ &= b^2 + \underbrace{icb - ibc}_{i(cb-bc)} + c^2 \end{aligned} \qquad \begin{aligned} a^*\,a &= \underbrace{(b-ic)}_{b^*-ic^*}(b+ic) \\ &= b^2 \underbrace{-icb + ibc}_{-i(cb-bc)} + c^2 \end{aligned}$$

By comparing these expressions, $a^*\,a = a\,a^*$ if and only if $cb - bc = 0$ or $cb = bc$

(6) If A has a unit $\mathbb{1}$ and x is invertible. Then

$$x^{-1}\,x = x\,x^{-1} = \mathbb{1}$$

then applying the involution,

$$x^*\,(x^{-1})^* = (x^{-1})^*\,x^* = \mathbb{1}^* = \mathbb{1}$$

Hence x^* has an inverse which can be identified as $(x^{-1})^*$.

(7) First, we note that for $x \in A$,

$$\|x\|^2 \leq \|x^*\,x\| \leq \|x^*\|\|x\|$$

So we know, $\|x\| \leq \|x^*\|$. Thus, implies that

$$\|x\| \leq \|x^*\| \leq \|(x^*)^*\| = \|x\|$$

Hence, equality holds throughout. Returning to the first chain of inequality gives

$$\|x\|^2 \leq \|x^*\,x\| \leq \|x^*\|\|x\| \leq \|x\|\|x\| = \|x\|^2$$

So the quality holds between $\|x\|^2$ and $\|x^*\,x\|$

□

2.5 Example (for C*-algebra). Let X be a locally Hausdorff space, $C_0(X)$ is the set of the continuous on X such that for each $\epsilon > 0$, there is a compact set K, if $x \notin K$, $|f(x)| < \epsilon$. We equip $C_0(X)$ with a norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)|$$

This is a closed subspace of the bounded, continuous functions on X. With $f^*(x) = \overline{f(x)}$, this becomes C*-algebra

Proof.

- Completeness:
  Consider a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_0(X)$. Then we have

  $$\|f_n - f_m\|_\infty = \sup_{x\in X} |f_n(x) - f_m(x)| \to 0 \text{ as } n, m \to \infty$$

  Define $f : X \to \mathbb{C}$ as $f(x) = \lim_{n\to\infty} f_n(x)$. Then $|f_n(x) - f(x)| \to 0$ as $n \to \infty$

  First, we show that $\lim_{n\to\infty} f_n = f$. We have that

  $$\begin{aligned}\|f - f_n\|_\infty = \sup_{x\in X} |f(x) - f_n(x)| &= \|f - f_n\|_\infty = \sup_{x\in X} |f(x) - f_m(x) + f_m(x) - f_n(x)| \\ &\leq \sup_{x\in X} \underbrace{|f(x) - f_m(x)|}_{\substack{\to 0 \\ \text{since } f(x) = \lim_{m\to\infty} f_m(x)}} + \underbrace{\sup_{x\in X} |f_m(x) - f_n(x)|}_{\substack{\xrightarrow{n,m\to\infty} 0 \\ \text{since } f_n \text{ is Cauchy}}} \\ &\longrightarrow 0 \qquad \text{as } n, m \to \infty\end{aligned}$$

  Next, we show that $f \in C_0(X)$.

  Since $f_n \in C_0(X)$, then f_n is continuous at $x \in X$. Given $\epsilon > 0$, there is $\delta > 0$ such that for all $y \in X$, $\|x - y\| < \delta$, implies that $|f_n(x) - f_n(y)| < \epsilon$. With $x \in X$ and same condition such that $\|x - y\| < \delta$, we have

  $$\begin{aligned}|f(x) - f(y)| &= |\lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(y)| \\ &\leq \lim_{n\to\infty} |f_n(x) - f_n(y)| < \epsilon\end{aligned}$$

  Thus, f is continuous on X. Hence $f \in C_0(X)$ because for each $\epsilon > 0$, there is a compact set K, such that if $x \notin K$, $|f(x)| = |\lim_{n\to\infty} f_n(x)| \leq \lim_{n\to\infty} |f_n(x)| < \epsilon$ (same compact set K in $f_n \in C_0(X)$ condition).

- Now, we show that $C_0(X)$ is an algebra. Let $f, g \in C_0(X)$, then

  $$\begin{aligned}\|f \cdot g\|_\infty = \sup_{x\in X} |f(x)g(x)| &\leq \sup_{x\in X} \left(\sup_{x\in X} |f(x)| \, |g(x)| \right) \\ &= \sup_{x\in X} |f(x)| \left(\sup_{x\in X} |g(x)| \right) = \|f\|_\infty \|g\|_\infty\end{aligned}$$

- Next, for $f \in C_0(X)$, $f^*(x) = \overline{f(x)}$. We have
$$\|f^*\|_\infty = \sup_{x\in X} |f^*(x)| = \sup_{x\in X} |\overline{f(x)}| = \sup_{x\in X} |f(x)| = \|f\|_\infty$$
- Finally, we show $C_0(X)$ is C^*-algebra . For each $f \in C_0(X)$, consider
$$f^* \cdot f(x) = f^*(x)\, f(x) = \overline{f(x)}\, f(x) = |f(x)|^2$$
taking sup over X, we get
$$\|f^* \cdot f\|_\infty = \sup_{x\in X} |f(x)|^2 \geq \left(\sup_{x\in X} |f(x)|\right)^2 = \|f\|_\infty^2$$

Hence by Lemma 2.3(7), this completes the proof. □

For this C^*-algebra , the map $\delta_x : C_0(X) \to \mathbb{C}$ with $f \mapsto f(x)$ is a (nontrivial) character on $C_0(X)$. This is because of Urysohn's lemma which guarantees the existence of a function $f \in C_0(X)$ with $f(x) = 1$. We will see later, $(\widehat{C_0(X)}) = \{\delta_x : \ x \in X\}$.

As a special example, if $X = \mathbb{N}$, $C_0(X) = c_0$ and $\hat{c_0} = \{\delta_n : \ n \in \mathbb{N}\} \cong \mathbb{N}$.

More examples with different types of norm.

2.6 Examples. Let S be an involutive semigroup. Consider $\ell^1(S)$ i.e. the space of all $f : S \to \mathbb{C}$ with $\|f\|_1 = \sum\limits_{s\in S} |f(s)| < \infty$. (Note that the set $\{s \in S : \ f(s) \neq 0\}$ is at most countable).

Equip $\ell^1(S)$ with the convolution
$$(f * g)(s) = \sum_{\substack{a,b\in S\\ ab=s}} f(a)g(b)$$
and let $f^*(s) = \overline{f(s^*)}$. Then $\ell^1(S)$ becomes a Banach-*-algebra .

Proof.

- First, we see that $\ell^1(S)$ is closed under convolution. Let $f, g \in \ell^1(S)$. Then
$$\|f\|_1 = \sum_{s\in S} |f(s)| < \infty \qquad \text{and} \qquad \|g\|_1 = \sum_{s\in S} |g(s)| < \infty$$
Let $J_f = \{s \in S : f(s) \neq 0\}$ and $J_g = \{s \in S : g(s) \neq 0\}$. Note that J_f and J_g are at most countable. Consider
$$\begin{aligned}\|f * g\|_1 &= \sum_{s\in S} |f * g(s)| = \sum_{s\in S} |\sum_{\substack{a,b\in S\\ ab=s}} f(a)g(b)| \\ &\leq \sum_{s\in S} \sum_{\substack{a,b\in S\\ ab=s}} |f(a)|\,|g(b)| = \sum_{a\in S} \left(|f(a)| \sum_{\substack{b\in S\\ ab=s}} |g(b)|\right) \qquad (1) \\ &\leq \sum_{a\in S} \Big(|f(a)|\ \|g\|_1\Big) = \|g\|_1\ \|f\|_1 < \infty\end{aligned}$$
Thus, $f * g \in \ell^1(S)$.

- Next, we show that $\ell^1(S)$ is a Banach algebra. Consider a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset \ell^1(S)$. Define $f : S \to \mathbb{C},\ f(s) = \lim_{n\to\infty} f_n(s)$.

$$\begin{aligned}\|f - f_n\|_1 &= \sum_{s\in S} |f(s) - f_n(s)| = \sum_{s\in S} |f(s) - f_m(s) + f_m(s) - f_n(s)| \\ &\leq \sum_{s\in S} \underbrace{|f(s) - f_m(s)|}_{\substack{\xrightarrow{\text{for each } s} 0 \\ \text{since } f(s)=\lim_{m\to\infty} f_m(s)}} + \underbrace{\sum_{s\in S} |f_m(s) - f_n(s)|}_{\substack{\xrightarrow{n,m\to\infty} 0 \\ \text{since } f_n \text{ is Cauchy}}} \\ &\longrightarrow 0 \qquad \text{as } n, m \to \infty\end{aligned}$$

Also,

$$\|f\|_1 = \sum_{s\in S} |f(s)| = \sum_{s\in S} |\lim_{n\to\infty} f_n(s)| \leq \lim_{n\to\infty} \sum_{s\in S} |f_n(s)| < \infty$$

Thus, $\{f_n\}_{n\in\mathbb{N}}$ converges to $f \in \ell^1(S)$. So $\ell^1(S)$ is a Banach space and we then even have $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ as in equation 1. Hence, it is a Banach algebra.

- And finally, we show $\ell^1(S)$ is a Banach-*-algebra . Let $f \in \ell^1(S)$ and $f^*(s) = \overline{f(s^*)}$. Then

$$\begin{aligned}\|f^*\|_1 &= \sum_{s\in S} |f^*(s)| = \sum_{s\in S} |\overline{f(s^*)}| = \sum_{s\in S} |f(s^*)| \\ &= \sum_{s^*\in S} |f(s)| = \|f\|_1\end{aligned}$$

□

We than have a homomorphism $\eta : S \to \ell^1(S)$ that maps $s \mapsto \delta_s$ with

$$\delta_s(t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{elsewhere} \end{cases}$$

For these,

$$\begin{aligned}(\delta_s * \delta_t)(x) &= \sum_{\substack{a,b\in S \\ ab=x}} \delta_s(a)\,\delta_t(b) \\ &= \begin{cases} 1 & \text{if } \delta_s(a) = 1 = \delta_t(b) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } s = a, t = b \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } ab = st = x \\ 0 & \text{otherwise} \end{cases} \\ &= \delta_{st}(x)\end{aligned}$$

Since

$$\begin{aligned}\delta_s^*(t) &= \overline{\delta_s(t^*)} = \delta_s(t^*) = \begin{cases} 1 & \text{if } s = t^* \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } s^* = (t^*)^* = t \\ 0 & \text{otherwise} \end{cases} \\ &= \delta_{s^*}(t)\end{aligned}$$

then $\delta_s^* = \delta_{s^*}$, we even have $\eta(s^*) = \delta_{s^*} = \delta_s^* = (\eta(s))^*$, so η is a homomorphism that identifies the involutive semigroup with a subset of $\ell^1(S)$, so it embeds S in the Banach-*-algebra .