Lecture Notes from October 11, 2022

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Last time:
- Properties of Banach - * -algebra
- Examples:

We had seen two examples of such concrete examples.
1) $C_0(X)$ (C* - algebra)
2) $\ell^1(S)$ (Banach - * - algebra)

Today, we will see few more examples.

Foreshadowing:

Recall, we had connected the Spectral Theory with The Representation Theory.

From finite dimension representations of commutative involutive semi-groups, we had deduced that each normal $A \in B$ (C^n) can be diagonalized.

The space in (C^n) was split in a direct sum of eigenspaces of A.

Consider now, instead, a locally compact Hausdroff Space $C_0(X)$ (we can replace this space with R if we want to). Let $A \in C_0(X)$ act on $l^2(X)$ by $(Af)(x) = A(x)f(x)$.

Then A is normal. (Here * is a conjugation on Banach algebra so, $AA^* = A^*A$)

2.8 Question. : Does $l^2(X)$ split into a direct sum of eigenspaces? Is l^2 is special case of C^n? Could we have written l^∞ and would it still be the same?

Answer: Yes, indeed this is the case because we have $\{\delta x : x \in X\}$ as canonical basis (orthonormal basis) of eigenvectors of A. So, then, if we want to take L^2 instead of l^2, there is a problem. We do not get orthonormal basis of eigenvectors easily because there are A's that do not work.
Example: Suppose $\delta_x = X = [0, 1]$.
Let A be given by $(A\delta_x)(f(x)) = A(x)\delta_x f(x) = A(x)f(x) = Af(x)$

Here, we know from previous chapters that A does not have eigenvectors.

Thus, each basis vector is an eigenvector because its 0 everywhere except at one point. This shows us that the orthonormal basis of eigenvectors are present in some cases but not in others and hence, motivates us to find something that generalizes the eigenvalues and eigenvectors. So, we will drive the insights we have for upcoming topics.

Goal: With examples with involutive algebras, we exhaust as much as possible what is out there. So, we will try to understand in some series.

We will continue with some more examples.

2.9 Example. : **Hermition Version**

Let the semigroup $S = \{N_0, +\}$ (non negative $\mathbb{Z}$ with $+$), involution $S^* = S$, then $\ell^1(S) \approx \ell^1$, and for x,y $\in \ell^1(S)$ $(x * y)_n = \sum_{k=0}^{n} x_k y_{n-k}$

Here, ' * ' of function is a complex conjugate.
By the cauchy-product, the function $\sum \ell^1 \to \mathbb{C}$,

$$\begin{aligned}
\sum(x) &= \sum_{n=0}^{\infty} X_n \\
gives\ \sum(x*y) &= \sum_{n=0}^{\infty}(x*y)_n \\
&= (x_0y_0) + (x_1y_0 + x_0y_1) + (x_2y_0 + x_1y_1 + x_0y_2) + ... \\
&= x_0(y_0 + y_1 + ...) + x_1(y_0 + y_1 + ...) + ... \\
&= \sum_{n=0}^{\infty}(x)_n \sum_{n=0}^{\infty}(y)_n \\
&= \sum(x)\sum(y) \\
and\ since\ \sum(x^*) &= \overline{\sum(x)},
\end{aligned}$$

we have, $\sum$ is a character of $\ell^1(S)$

Now, we want to describe all characters of $\ell^1(S)$. At this point we know $\ell^1(S)$ is algebra with involution and also a Banach - * - algebra but not C^*-algebra yet with respect to this norm.

Whenever we have norm, the characters need to be bounded linear functional with non-zero homomorphism.

Lastly, Let $\mathcal{X} : \ell^1(S) \to B(\mathbb{C}) = \mathbb{C}$ be bounded linear character. Then, we also note $\eta : S \to \ell^1(S)$. Here, S is the semi group itself and η is 0 everywhere except position S and $\eta(S) = \delta s$

Then, precomposing character with S, $\mathcal{X}$o $\eta : S \to \mathbb{C}$ is a character on S.

$$\begin{aligned}
&\text{Let } x, y \in S \text{ and } \phi = \mathcal{X}\text{o } \eta : S \to \mathbb{C}. \\
&\text{Here, } \phi(s) = \mathcal{X}\text{o } \eta(s) = \mathcal{X}(\eta(s)) = \mathcal{X}(\delta s), \phi(t) = \mathcal{X}\text{o } \eta(t) = \mathcal{X}(\eta(t)) = \mathcal{X}(\delta t) \\
\text{Now, } (\phi(s))(\phi(t)) &= (\mathcal{X}(\delta s))(\mathcal{X}(\delta t)) \\
&= (\mathcal{X}(\delta s)(\delta t)) \\
&= (\mathcal{X}(\delta st)) \\
&= \mathcal{X}\text{o } \eta(st) \\
&= (\phi(st))
\end{aligned}$$

By doing this, we embedded δ into $l^1(S)$. Notice, if we have character if $l^1(S)$, it passes to S itself, but embedding preserves structure of S.

We want to see if we can reconstruct character by what it does on S.

By span $\{\eta(S)\}$ being dense in $l^1(S)$, $\mathcal{X}$ continuous, for $f \in l^1(S)$,

$$\begin{aligned}
\mathcal{X}(f) &= \mathcal{X}(\sum_{S \in S} f(S)\eta(S)) \\
&= \sum_{S \in S} f(S)\mathcal{X}(\eta(S))
\end{aligned}$$

So since the dual space of $l^1(S)$ is $l^\infty(S)$. Here, we can note that $\mathcal{X}(\eta(S))$ cannot give sequence that is possibly unbounded. So, $\mathcal{X}\text{o}\eta : S \to \mathbb{C}$ is necessarily bounded.

Conversely , each bounded character $r : S \to \mathbb{C}$ defined by, $\mathcal{X}(f) = \sum f(S)r(S)$ a bounded character on $l^1S)$.

Hence, we have $(l^1(S)\hat{)} \approx \hat{S} \cap l^\infty(S)$

(* Characters on $l^1(S)$ are precisely given by characters on S, but δ is the non-neg $\mathbb{Z}$ so $\hat{S}$ here is $\mathcal{R}/0$)

Our $\mathbf{Upshot}$ here is that In our case, $S = \mathbb{N}_0$

and, $(l^1(S))\hat{}_0$ = [-1, 1]

(Here, values of characters at element 1 and $\sum$ is sum over all elements that produces 1 and 0 sitting on $(l^1(S))_0$ adds a little extra complication.)

2.10 Example. **Weiner Algebra**

Consider $l^1(\mathbb{Z})$, and let for x $\in l^1(\mathbb{Z})$,

$$(X^*)_n = \overline{X_{-n}}$$
$$and\ (x * y)_n = \sum_{k \in \mathbb{Z}} x_k y_{n-k}$$

We will see that we can think of this algebra as an algebra of functions on the unit circle.

We can write this algebra via the Fourier transform.

$$F(x)(z) = \sum_{n \in \mathbb{Z}} x_n z^n,\ |z| = 1$$
$$then\ |F(x)(z)| \le ||x||_1$$
$$< \infty$$

By Weierstrass, the series converges uniformly on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ to a continuous function.

Let us abbreviate $\hat{x}$ (z) = $\sum_{n \in \mathbb{Z}} x_n z^n$ is the F for fourier trasform.

l^1 maps over to space of continuous function becomes C*-algebra

We recall $C(S^1)$ is a C*-algebra with $f^*(z)$ = f(z), |z|=1, and using pointwise multiplication as product.

Map of Fourier transform takes l^1 to $C(S^1)$

We claim F: $l^1(\mathbb{Z}) \to C(S^1)$ is a homomorphism of Banach *-algebra.

We know, $||\hat{x}||_\infty \le ||x||_1$

F is bounded linear map. Next , we focus on structure.

Furthermore,

$$\begin{aligned}
\hat{x^*}(z) &= \sum_{n\in\mathbb{Z}} \overline{x_{-n}} z^n \quad \text{(Here } \hat{x^*}(z) \text{ is embedded function on unit circle)} \\
&= \sum_{n\in\mathbb{Z}} \overline{x_n} z^{-n} \quad \text{(Replacing n by -n since n is a dummy sequence)} \\
&= \overline{\sum_{n\in\mathbb{Z}} x_n z^n} \quad \text{by linearity and summabitlity} \\
&= (\hat{x})^*(z) \quad \text{(involution stays involution)}
\end{aligned}$$

Moreover,

$$\begin{aligned}
\hat{x}(z)\hat{y}(z) &= \sum_{n\in\mathbb{Z}} x_n z^n \sum y_m z^m \quad \text{(we took pointwise multiples as product)} \\
&= \sum_{n\in\mathbb{Z}} [\sum_{k\in\mathbb{Z}} x_k y_{n-k}\ ] z^n \quad \text{(Here, shifted index k = n+m)} \\
&= (x * y)\hat{}(z)
\end{aligned}$$

(Here, the map F is invertible on its range and sends all of $l^1(z)$ inside $C(S^1)$).
The map F is invertible (on its range) because given any f : $S^1 \to \mathbb{C}$, f is continuous, we consider $\check{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}dt$

For x $\in l^1(\mathbb{Z})$, we can compute f = $\hat{x}$, we note that

$$\begin{aligned}
\check{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} (\sum_{m\in\mathbb{Z}} x_m e^{imt})e^{-int}dt \quad \text{(Here, integration is 0 except when m = M)} \\
&= x_n
\end{aligned}$$

because summation can be exchanged with integration.

Here, $\hat{x}$ is the Fourier series.

Is this onto?

It turns out that $C(S^1)$ is actually bigger than image of $l^1(\mathbb{Z})$ sitting inside $C^1(S)$.
Thus, it has a special name "Weiner-algebra."

(We know that inverse which is normal is point-wise inverse but it does not guarantee its boundedness. Thus, if we have a point-wise inverse, it is unbounded. Likewise, if the point-wise exists, then we have a bounded function.)

Conversely, for any $f \in C(S^1)$, $\check{f}$ may not be in $l^1(\mathbb{Z})$

We conclude, F maps $l^1(\mathbb{Z})$ to a proper subalgebra of $C(S^1)$ but the Fourier series does not map onto $C(S^1)$, because there are continuous functions whose Fourier coefficients are not summable.