Lecture Notes from October 11, 2022

taken by Cristian Meraz

Last time

- Properties of Banach $*$-algebras.
- Examples: $C_0(X)$ and $\ell^1(S)$.

1.0.1 Warm-up

Recall from finite dimensional representations of commutative involutive semigroups, we had deduced that each normal $A \in B(\mathbb{C}^n)$ can be diagonalized, namely, splits into a direct sum of eigenspaces of A.
Consider now, instead, the space $C_0(X)$, and let $A \in C_0(X)$ act on $\ell^2(X)$ by

$$(Af)(x) = A(x)f(x).$$

Recall that for $A \in C_0(X)$ we have $A : X \to \mathbb{C}$ and the involution is given by $A^*(x) = \overline{A(x)}$. Then A is normal because $AA^* = A^*A$ as in complex numbers, and

$$\begin{aligned}(AA^*f)(x) &= A(\overline{A(x)}f(x)) = A(x)\overline{A(x)}f(x) \\ &= \overline{A(x)}A(x)f(x),\end{aligned}$$

hence, $(AA^*f)(x) = (A^*Af)(x)$.

1.6 Question. Does $\ell^2(X)$ split into a direct sum of eigenspaces?

1.7 Answer. Yes, we have $\{\delta_x : x \in X\}$ as an orthonormal basis of eigenvectors of A, where

$$\delta_x(y) = \begin{cases} 1, & y = x, \\ 0, & \text{otherwise.} \end{cases}$$

We have discussed such functions as an orthonormal basis of $\ell^2([0,1])$. To see they are eigenvectors of A, note

$$\begin{aligned}(A\delta_x)(f(x)) &= A(x)\delta_x(x)f(x) \\ &= A(x)f(x) \\ &= (Af)(x)\end{aligned}$$

for each $x \in X$.

1.0.2 More examples

1.8 Example (Hermitian case). Let $S = (\mathbb{N}_0, +)$, $s^* = s$, then $\ell^1(S) \simeq \ell^1$, and for each $x, y \in \ell^1(S)$,

$$(x * y)_n = \sum_{k=0}^{n} x_k y_{n-k}.$$

By the Cauchy product (i.e., discrete convolution 'product'),

$$\Sigma : \ell^1(S) \to \mathbb{C},\ \Sigma(x) = \sum_{n=0}^{\infty} x_n,$$

gives

$$\Sigma(x * y) = \Sigma(x)\Sigma(y).$$

Computing the first few terms, we see that

$$\begin{aligned}
\Sigma(x * y) &= \sum_{n=0}^{\infty} (x * y)_n \\
&= \underbrace{x_0 y_0}_{n=0} + \underbrace{x_0 y_1 + x_1 y_0}_{n=1} + \cdots + \underbrace{x_0 y_k + x_1 y_{k-1} + \cdots + x_{k-1} y_1 + x_k y_0}_{n=k} + \cdots \\
&= x_0(y_0 + y_1 + \cdots + y_k + \cdots) + x_1(y_0 + y_1 + \cdots y_k + \cdots) + \cdots + x_k(y_0 + \cdots) + \cdots \\
&= x_0 \left(\sum_{n=0}^{\infty} y_n \right) + \cdots + x_k \left(\sum_{n=0}^{\infty} y_n \right) + \cdots \\
&= \left(\sum_{n=0}^{\infty} x_n \right) \left(\sum_{n=0}^{\infty} y_n \right) = \Sigma(x)\Sigma(y).
\end{aligned}$$

Thus, Σ is a homomorphism. Moreover, since $\Sigma(x^*) = \overline{\Sigma(x)}$, Σ is a character of $\ell^1(S)$. Our goal is to describe all characters of $\ell^1(S)$.

To this end, let

$$\chi : \ell^1(S) \to \mathbb{C}$$

be a (bounded) character, and consider

$$\eta : S \to \ell^1(S),\ \eta(s) = \delta_s,$$

where $\delta_s \in \ell^1(S)$ is the element that is one at position s and zero elsewhere. In this way we embed S into $\ell^1(S)$. Then

$$\chi \circ \eta : S \to \mathbb{C}$$

is a character on S. To see this, let $s, t \in S$, and write $\psi = \chi \circ \eta : S \to \mathbb{C}$. Then $\psi(s) = (\chi \circ \eta)(s) = \chi(\delta_s)$ and $\psi(t) = (\chi \circ \eta)(t) = \chi(\delta_t)$, so that

$$\begin{aligned}
\psi(s)\psi(t) &= \chi(\delta_s)\chi(\delta_t) \\
&= \chi(\delta_s \delta_t) \quad (\chi \text{ is a homomorphism on } \ell^1(S)) \\
&= \chi(\delta_{st}) \quad (\delta \text{ is a homomorphism on } S) \\
&= (\chi \circ \eta)(st) \\
&= \psi(st).
\end{aligned}$$

Moreover, by Span$\{\eta(s)\}$—the functions of finite support— being dense in $\ell^1(S)$, and χ continuous, for $f \in \ell^1(S)$, we have

$$\begin{aligned}\chi(f) &= \chi\left(\sum_{s\in S} f(s)\eta(s)\right)\\ &= \sum_{s\in S} f(s)\chi(\eta(s)),\end{aligned}$$

Since the dual space of $\ell^1(S)$ is $\ell^\infty(S)$, we note that $\chi \circ \eta : S \to \mathbb{C}$ is necessarily bounded.

Conversely, each bounded character $\gamma : S \to \mathbb{C}$ defines by

$$\chi(f) = \sum_{s\in S} f(s)\gamma(s)$$

a (bounded) character on $\ell^1(S)$, hence we have

$$\widehat{(\ell^1(S))} = \hat{S} \cap \ell^\infty(S).$$

The upshot is: In our case, $S = \mathbb{N}_0$, and

$$\widehat{(\ell^1(S))}_0 \simeq [-1, 1].$$

1.9 Example (Wiener algebra). Consider $\ell^1(\mathbb{Z})$, and let, for $x \in \ell^1(\mathbb{Z})$,

$$(x^*)_n = \overline{(x_{-n})}, \quad \text{and} \quad (x * y)_n = \sum_{k\in\mathbb{Z}} x_k y_{n-k}.$$

We can view this algebra via the Fourier transform,

$$F(x)(z) = \sum_{n\in\mathbb{Z}} x_n z^n, \quad |z| = 1,$$

then $|F(x)(z)| \leq \|x\|_1 < \infty$, and the series converges uniformly on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ to a continuous function. Engineers call it the z-transform.

Let us abbreviate

$$\hat{x}(z) = \sum_{n\in\mathbb{Z}} x_n z^n.$$

We recall $C(S^1)$ is a C^*-algebra with $f^*(z) = \overline{f(z)}$, $|z| = 1$, and using pointwise multiplication as product. In particular, the Fourier transform maps ℓ^1 into $C(S^1)$.

1.10 Claim. The Fourier transform map $F : \ell^1(\mathbb{Z}) \to C(S^1)$ is a homomorphism of Banach $*$-algebras.

Proof. We know that $\|\hat{x}\| \leq \|x\|_1$, therefore F is bounded and linear. Furthermore, write

$$\widehat{x^*(z)} = \sum_{n\in\mathbb{Z}} \overline{x_{-n}} z^n.$$

Realizing n is just a dummy variable, let $n' = -n$ and substitute

$$\begin{aligned}
\widehat{x^*(z)} &= \sum_{n' \in \mathbb{Z}} \overline{x_{-n'}} z^{n'} \\
&= \sum_{n \in \mathbb{Z}} \overline{x_n} z^{-n} \\
&= \sum_{n \in \mathbb{Z}} \overline{x_n z^{-n}} \\
&= (\hat{x})^*(z).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\hat{x}(z)\hat{y}(z) &= \sum_{n \in \mathbb{Z}} x_n z^n \sum_{m \in \mathbb{Z}} y_m z^m \\
&= \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} x_k y_{n-k} \right) z^n \\
&= \widehat{(x * y)}(z).
\end{aligned}$$

□

The map F is invertible on its range, because given any $f : S^1 \to \mathbb{C},$ f continuous, we consider

$$\check{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} \, dt.$$

For $x \in \ell^1(\mathbb{Z})$ we can compute $f = \hat{x}$, and we note that

$$\check{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{m \in \mathbb{Z}} x_m e^{imt} \right) e^{-int} \, dt = x_n,$$

because summation and integration can be interchanged.

1.11 Question. Is this map onto?

1.12 Answer. No, $C(S^1)$ is larger than the image of $\ell^1(\mathbb{Z})$ inside of $C(S^1)$. Conversely, for any $f \in C(S^1)$, $\check{f}$ may not be in $\ell^1(\mathbb{Z})$. We conclude that F maps $\ell^1(\mathbb{Z})$ to a proper subalgebra of $C(S^1)$; we call it the Wiener algebra.