Lecture Notes from October 13, 2022

taken by Anshi Gupta

Last Time
Banach$-*-$ algebras and characters by examples.
Warm up: Consider $l^1(\mathbb{Z})$ as Banach$-*-$ algebra with $x * y$ defined by convolution and $(x^*)_k = \overline{(x_{-k})}$ for $x \in l^1(\mathbb{Z}), k \in \mathbb{Z}$.

4.5 Question. What are the characters on this Banach$-*-$ algebra?

Again, we use $\eta : k \to \delta_k$, then for $f \in l^1(\mathbb{Z})$, a (bounded) character χ,

$$\begin{aligned}\chi(f) &= \chi(\sum_{k\in\mathbb{Z}} f(k)\eta(k)) \\ &= \sum_{k\in\mathbb{Z}} f(k)\chi(\eta(k))\end{aligned}$$

and by $\|\chi\| < \infty, (l^1(\mathbb{Z}))' = l^\infty(\mathbb{Z})$, we know that $\chi\circ\eta$ is bounded and $\chi\circ\eta$ is a character on $\mathbb{Z}$ from the embedding of $\mathbb{Z}$ in the algebra.Hence, determined by $\chi\circ\eta(1) = z \in S^1 = \{w : |w| = 1\}$. Conversely, if γ is a character on $\mathbb{Z}$, then $\chi(f) = \sum_{k\in\mathbb{Z}} f(k)\gamma(k)$ defines a character on $l^1(\mathbb{Z})$. We summarize, $\widehat{l^1(\mathbb{Z})} \cong \hat{\mathbb{Z}} \cong S^1$.

Consequently, we can map $l^1(\mathbb{Z})$ to a space in $\mathbb{C}(S^1)$, using that each character γ of $\mathbb{Z}$ is of the form $k \mapsto \mathbb{Z}^k$, hence we can define $\hat{f}(z) = \sum_{k\in\mathbb{Z}} f(k)z^k$.
Next, we will see that we will see that we can relate Banach$-*-$ Algebras with the Fourier transform in a similar way.

4.6 Example. We first construct a algebra with involution. Let $C_c(\mathbb{R}^n)$ be the space of continuous functions with compact support. Define $f^*(x) = \overline{f(-x)}$ and for $x \in \mathbb{R}^n; f, g \in C_c(\mathbb{R}^n); (f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)d\lambda(y)$, where λ is a Lebesgue measure. For such f and g, $f * g$ is again in $C_c(\mathbb{R}^n)$. This is because, since g is continuous so for given $\epsilon > 0, \exists\delta > 0$ such that $\|x-y\| < \delta \implies |g(x) - g(y)| < \epsilon$.
Now, f is continuous on a compact set, this implies f is bounded. Therefore, $\exists M > 0$ such that

$\|f(x)\| < M$. Consider,

$$\begin{aligned}\|f*g(x) - f*g(y)\| &= \|\int_{\mathbb{R}^n} f(z)[g(x-z) - g(y-z)]d\lambda(z)\| \\ &\leq \int_{\mathbb{R}^n} \|f(z)\|\|[g(x-z)-g(y-z)]\|d\lambda(z) \\ &\leq M\epsilon \int_{\mathbb{R}^n} d\lambda(z) \\ &= M\epsilon\lambda(\mathbb{R}^n)\end{aligned}$$

since g being continuous and compactly supported is uniformly continuous. Therefore, $f * g$ is continuous. Also, $Supp(f*g) \subset Supp(f^*) \cup Supp(g^*)$, and since union of two compact sets is compact, $f*g$ is compact. Hence, $f*g \in C_c(\mathbb{R}^n)$. This space forms a commutative involutive algebra.
We show associativity,

$$\begin{aligned}((f*g)*h)(x) &= \int_{\mathbb{R}^n} f(x*y)(y)h(x-y)d\lambda(y) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z)g(y-z)d\lambda(z)h(x-y)d\lambda(y) \\ &\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} f(z)\int_{\mathbb{R}^n} g(y-z)h(x-y)d\lambda(y)d\lambda(z) \\ &\overset{y-z=u}{=} \int_{\mathbb{R}^n} f(z)\int_{\mathbb{R}^n} g(u)h(x-u-z)d\lambda(u)d\lambda(z) \\ &= \int_{\mathbb{R}^n} f(z)g*h(x-z)d\lambda(z) \\ &= (f*(g*h))(x)\end{aligned}$$

Moreover,

$$\begin{aligned}(f*g)*(x) &= \overline{\int_{\mathbb{R}^n} f(y)g(x-y)d\lambda(y)} \\ &= \int_{\mathbb{R}^n} \overline{f(y)g(-x-y)}d\lambda(y) \\ &\overset{u=-y}{=} \int_{\mathbb{R}^n} \overline{f(-u)g(-x+u)}d\lambda(u) \\ &= \int_{\mathbb{R}^n} f^*(u)g^*(x-u)d\lambda(u) \\ &= (f^* * g^*)(x)\end{aligned}$$

Together with,

$$\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}^n} f(y)g(y-x)d\lambda(y) \\
&\overset{u=-y}{=} \int_{\mathbb{R}^n} f(-u)g(x+u)d\lambda(u) \\
&\overset{w=x+u}{=} \int_{\mathbb{R}^n} g(w)f(x-w)d\lambda(w) \\
&= (g * f)(x)
\end{aligned}$$

Hence, $C_c(\mathbb{R}^n)$ forms an involutive algebra.

Next, we define, $\|f\|_1 = \int_{\mathbb{R}^n} |f(x)|d\lambda(x)$ We claim this norm is sub - multiplicative. To see this, consider $f, g \in C_c(\mathbb{R}^n)$

$$\begin{aligned}
\|f * g\|_1 &= \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} f(y)g(x-y)d\lambda(y)|d\lambda(x) \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)||(x-y)|d\lambda(y)d\lambda(x) \\
&\overset{\text{Fubni}}{=} \int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} |g(x-y)|d\lambda(x)d\lambda(y) \\
&= \int_{\mathbb{R}^n} |f(y)|\|g\|_1 d\lambda(y) \\
&= \|g\|_1 \int_{\mathbb{R}^n} |f(y)|d\lambda(y) \\
&= \|g\|_1\|f\|_1
\end{aligned}$$

We also observe, $\|f^*\|_1 = \|f\|_1$. Since

$$\begin{aligned}
\|f^*\|_1 &= \int_{\mathbb{R}^n} |\overline{f(-x)}|d\lambda(x) \\
&= \int_{\mathbb{R}^n} |f(-x)|d\lambda(x) \\
&\overset{u=-x}{=} \int_{\mathbb{R}^n} |f(u)|d\lambda(u) \\
&= \|f\|_1
\end{aligned}$$

Now, taking $L^1(\mathbb{R}^n)$ to be the completion of $C_c(\mathbb{R}^n)$, then by continuity of $L^1(\mathbb{R}^n)$, we can extend $f * g$ and $f \to f^*$ to $L^1(\mathbb{R}^n)$ since if f_n and g_n are two Cauchy sequences in $C_c(\mathbb{R}^n)$ such that $f_n \to f$ and $g_n \to g$ in $L^1(\mathbb{R}^n)$ and g_n is continuous on $B(0, r)$ for some $r > 0$, we have

$$\begin{aligned}
\|f_n * g_n - f_m * g_m\|_1 &= \|\int_{\mathbb{R}^n} f_n(y)g_n(x-y)d\lambda(y) - \int_{\mathbb{R}^n} f_m(y)g_m(x-y)d\lambda(y)\|_1 \\
&\leq \int_{\mathbb{R}^n} \|f_n(y) - f_m(y)\|\|g_n(x-y) - g_m(x-y)\|_1 d\lambda(y)
\end{aligned}$$

This implies $\|f_n * g_n - f_m * g_m\|_1 \to 0$ as $n \to \infty$ as f_n and g_n are Cauchy. Hence, $f_n * g_n$ is Cauchy and convergent in $L^1(\mathbb{R}^n)$. Also, $f_n * g_n$ is uniformly continuous on $B(0,r)$ since it is continuous and compactly supported. Restricting the the convolution to B(0,r), we get

$$\begin{aligned}\|f_n * g_n - f * g\|_1 &= \|\int_{B(0,r)} f_n(y) g_n(x-y) d\lambda(y) - \int_{B(0,r)} f(y) g(x-y) d\lambda(y)\|_1 \\ &\leq \int_{B(0,r)} \|f_n(y) - f(y)\| \|g_n(x-y) - g(x-y)\|_1 d\lambda(y)\end{aligned}$$

Since, $f_n \to f$ and $g_n \to g$, we have $f_n * g_n \to f * g$ on $B(0,r)$. It converges on $L^1(\mathbb{R}^n)$, by taking the union of all the balls of radius $r > 0$ and,we obtain a Banach $-*-$ algebra. This algebra is also called the L^1- algebra of $\mathbb{R}^n$.
Next, we want to study the characters of this algebra. We consider an example

4.7 Example.

$$\begin{aligned}\chi_x : \mathbb{R}^n &\to S^1 \\ y &\mapsto e^{\iota x y}\end{aligned}$$

Then, χ_x is a continuous non trivial group homomorphism from $\mathbb{R}^n$ to S^1, hence a character on $\mathbb{R}^n$. By boundedness of χ_x, we obtain $\tilde{\chi_x} = \int_{\mathbb{R}^n} f(y) e^{\iota x.y} d\lambda(y)$ and we claim $\tilde{\chi_x}$ defines a character on $L^1(\mathbb{R})$. Indeed,

$$\begin{aligned}\tilde{\chi_x}(f^*) &= \int_{\mathbb{R}^n} \overline{f(-y)} e^{\iota x.y} d\lambda(y) \\ &\overset{u=-y}{=} \int_{\mathbb{R}^n} \overline{f(u)} e^{-\iota x.u} d\lambda(u) \\ &= \int_{\mathbb{R}^n} \overline{f(u) e^{\iota x.u} d\lambda(u)} \\ &= \overline{\tilde{\chi_x}(f)}\end{aligned}$$