Lecture Notes from October 13, 2022

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Last Time

- Example of Banach $*$-algebra and it's characters.

Warm up:(Finishing our previous example) Let $l^1(\mathbb{Z})$, and let for $x \in \ell^1(\mathbb{Z})$, $k \in \mathbb{Z}$, $(x^*)_n = \overline{(x_{-n})}$ and $(x * y) = \sum_{n\in} x_n y_{n-k}$ is a Banach $*$-algebra.

2.51 Question. what the characters on the Banach $*$-algebra $l^1(\mathbb{Z})$?

Let $\chi : l^1(\mathbb{Z}) \to \mathbb{C}$ be a (bounded) character, and consider $\eta : \mathbb{Z} \to l^1(\mathbb{Z})$, where $k \mapsto \delta_k$ is an embedding from $\mathbb{Z}$ to $l^1(\mathbb{Z})$. Then for $f \in l^1(\mathbb{Z})$,

$$\chi(f) = \chi(\sum_{k\in\mathbb{Z}} f(k)\eta(k)) = \sum_{k\in\mathbb{Z}} f(k)\chi(\eta(k))$$

and by $\|x\| < \infty$, $(l^1(\mathbb{Z}))' = l^\infty(\mathbb{Z})$, we know $\chi \circ \eta$ is a character on $\mathbb{Z}$ from the embedding of $\mathbb{Z}$ in the algebra, hence determined by

$$\chi \circ \eta(1) = z \in S^1 = \{\omega \in \mathbb{C} : |\omega| = 1\}$$

Conversely, if $\gamma.$ is a character on $\mathbb{Z}$, then

$$\chi(f) = \sum_{k\in\mathbb{Z}} f(k)\gamma(k)$$

defines a character on $l^1(\mathbb{Z})$.
We summarize, $\widehat{(l^1(\mathbb{Z}))} \cong \widehat{\mathbb{Z}} \cong S^1$.
Consequently, we can map $l^1(\mathbb{Z})$ to a space in $C(S^1)$, using that each character γ on $\mathbb{Z}$ is of the form $k \mapsto z^k$ where $\|z\| = 1$, hence

$$\chi(f)(k) = \widehat{f}(k) = \sum_{k\in\mathbb{Z}} f(k)z^k.$$

Next we will see that we can relate the Banach $*$-algebra with the fourier transform in a similar way.

2.52 Example. **We first construct an algebra with involution**

Let $C_c(\mathbb{R}^n)$ be the space of continuous functions with compact support.
Define $f^*(x) = \overline{f(-x)}$ and for $x \in \mathbb{R}^n$, $f, g \in C_c(\mathbb{R}^n)$

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)d\lambda(y)$$

where λ is the lebesgue measure. For such f, g, $f*g \in C_c(\mathbb{R}^n)$. Let us first look at the continuity of $f * g$. Since g is continuous, so for any sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^n$ such that $x_n \to x$, we have $(x_n - y) \to (x-y) \forall y \in \mathbb{R}^n \implies g(x_n - y) \to g(x-y)$ and also $|f(x)| \leq M$ for some $M > 0, \forall x \in \mathbb{R}^n$ since f is continuous on a compact set. Now consider,

$$\begin{aligned}
|f * g(x_n) - f * g(x)| &= |\int_{\mathbb{R}^n} f(y)[g(x_n - y) - g(x-y)]d\lambda y| \\
&\leq \int_{\mathbb{R}^n} |f(y)||g(x_n - y) - g(x-y)|d\lambda y \\
&\leq M \int_{\mathbb{R}^n} |g(x_n - y) - g(x-y)|d\lambda(y)
\end{aligned}$$

Therefore $|f * g(x_n) - f * g(x)| \to 0$ as $n \to \infty$ which shows that $f * g$ is continuous. Moreover, $f * g$ has compact support as $supp(f * g) = supp(f) + supp(g)$ where $supp(f)$ and $supp(g)$ are compact.
Claim: This space forms a commutative involutive algebra.
We show associativity,

$$\begin{aligned}
((f * g) * h)(x) &= \int_{\mathbb{R}^n} (f * g)(y)h(x-y)d\lambda(y) \\
&= \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} f(z)g(y-z)d\lambda(z)]h(x-y)d\lambda(y) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(z)g(y-z)h(x-y)d\lambda(z)\lambda(y) \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} f(z) \int_{\mathbb{R}^n} g(y-z)h(x-y)d\lambda(y)d\lambda(z) \\
&\overset{y-z=u}{=} \int_{\mathbb{R}^n} f(z) \int_{\mathbb{R}^n} g(u)h(x-u-z)d\lambda(u)d\lambda(y)d\lambda(z) \\
&= \int_{\mathbb{R}^n} f(z)g * h(x-z)d\lambda(z) \\
&= (f * (g * h))(x)
\end{aligned}$$

Moreover,

$$
\begin{aligned}
(f * g)^*(x) &= \overline{f * g}(-x) \\
&= \int_{\mathbb{R}^n} \overline{f(y)g(-x-y)} d\lambda(y) \\
&\overset{u=-y}{=} \int_{\mathbb{R}^n} \overline{f(-u)g(-x+u)} d\lambda(u) \\
&= \int_{\mathbb{R}^n} \overline{f(-u)g(-(x-u))} d\lambda(u) \\
&= \int_{\mathbb{R}^n} f^*(u)g^*(x-u) d\lambda(u) \\
&= (f^* * g^*)(x)
\end{aligned}
$$

together with

$$
\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}^n} f(y)g(x-y) d\lambda(y) \\
&\overset{u=-y}{=} \int_{\mathbb{R}^n} f(-u)g(x+u) d\lambda(u) \\
&\overset{\omega=x+u}{=} \int_{\mathbb{R}^n} g(\omega)f(x-\omega) d\lambda(\omega) \\
&= (g * f)(x)
\end{aligned}
$$

Therefore, we have

$$
(f * g)^*(x) = (f^* * g^*)(x) = (g^* * f^*)(x)
$$

Hence, $C_c(\mathbb{R}^n)$ forms an involutive algebra.

Next, we define $\|f\|_1 = \int_{\mathbb{R}^n} |f(x)| d\lambda(x)$. **Claim:** This norm is sub-multiplicative.
To see this, consider $f, g \in C_c(\mathbb{R}^n)$,

$$
\begin{aligned}
\|f * g\|_1 &= \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} f(y)g(x-y) d\lambda(y)| d\lambda(x) \\
&\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)||g(x-y)| d\lambda(y) d\lambda y \\
&\overset{\text{fubini}}{=} \int_{\mathbb{R}^n} f(y)|g(x-y)| d\lambda(x) \lambda(y) \\
&\overset{u=x-y}{=} \|g\|_1 \|f\|_1
\end{aligned}
$$

We also observe,

$$\begin{aligned}
\|f^*\|_1 &= \int_{\mathbb{R}^n} |f^*(x)| d\lambda(x) \\
&= \int_{\mathbb{R}^n} |\overline{f(-x)}| d\lambda(x) \\
&= \int_{\mathbb{R}^n} |f(-x)| d\lambda(x) \\
&\overset{u=-x}{=} \int_{\mathbb{R}^n} |f(u)| d\lambda(u) \\
&= \|f\|_1
\end{aligned}$$

Now taking $L^1(\mathbb{R}^n)$ to be the completion of $C_c(\mathbb{R}^n)$,then by convexity of $L^1(\mathbb{R}^n)$, we can extend $f * g$ and $f \to f^*$ to $L^1(\mathbb{R}^n)$ since if $f_n, g_n \in C_c(\mathbb{R}^n)$ such that $f_n \to f$ in $L^1(\mathbb{R}^n)$ and $g_n \to g$ in $L^1(\mathbb{R}^n)$, then since g_n's are continuous on $\overline{B(0,R)}$ for some $R > 0$, we have that

$$\begin{aligned}
\|f_n * g_n - f_m * g_m\|_1 &= \| \int_{\mathbb{R}^n} f_n(y) g_n((x-y) d\lambda(y) - \int_{\mathbb{R}^n} f_m(y) g_m(x-y) d\lambda(y)\|_1 \\
&\leq \int_{\mathbb{R}^n} \|f_n(y) - f_m(y)\|_1 \|g_n(x-y) - g_m(x-y)\|_1 d\lambda(y)
\end{aligned}$$

So, $\|f_n * g_n - f_m * g_m\|_1 \to 0$ as $n \to \infty$ since f_n, g_n are cauchy and therefore, $f_n * g_n$ is cauchy hence convergent in $L^1(\mathbb{R}^n)$. Also we know that $f_n * g_n$ being continuous is uniformly continuous in $\overline{B(0,R)}$. Then restricting the convolution on $\overline{B(0,R)}$, we have

$$\begin{aligned}
\|f_n * g_n - f * g\|_1 &= \| \int_{\overline{B(0,R)}} f_n(y) g_n((x-y) d\lambda(y) - \int_{\overline{B(0,R)}} f(y) g(x-y) d\lambda(y)\|_1 \\
&\leq \int_{\overline{B(0,R)}} \|f_n(y) - f(y)\|_1 \|g_n(x-y) - g(x-y)\|_1 d\lambda(y)
\end{aligned}$$

Hence, $\|f_n * g_n - f * g\|_1 \to 0$ on $\overline{B(0,R)}$. Thus, $f_n * g_n \to f * g$ on $L^1(\mathbb{R}^n)$ by taking the union of all the closed balls $\overline{B(0,R)}$ for $R > 0$ And we obtain a Banach $*-$algebra. This algebra is also called the L^1-algebra of $\mathbb{R}^n$.

Next, we want of study the characters of this algebra.
We consider an example, $\chi_x : \mathbb{R}^n \to S^1$ given by, $y \mapsto e^{ix.y}$.Then χ_x is a continuous non-trivial group homomorphism from $\mathbb{R}^n \mapsto S^1$. Hence a character on $\mathbb{R}^n$. By boundedness of χ_x, we obtain

$$\widetilde{\chi_x}(f) = \int_{\mathbb{R}^n} f(y) e^{ix.y} d\lambda(y)$$

We claim that $\widetilde{\chi_x}$ defines a character on $L^1(\mathbb{R}^n)$.

It's easy to see that $\widetilde{\chi_x}$ is linear by linearity of integrals. Also, $\widetilde{\chi_x}$ is non trivial as χ_x is so.

Next, we consider

$$\begin{aligned}
\widetilde{\chi_x}((f*g)) &= \int_{\mathbb{R}^n} (f*g)(y)e^{ix.y}d\lambda(y) \\
&= \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} f(z)g(y-z)d\lambda(z)]e^{ix.y}d\lambda(y) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z)e^{ix.y}g(y-z)e^{ix.(z-y)}d(z)d\lambda(y) \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} f(y)e^{ix.y}\int_{\mathbb{R}^n} g(z-y)e^{ix.(z-y)}d(z)d(y) \\
&= \int_{\mathbb{R}^n} f(y)e^{ix.y}\widetilde{\chi_x}(g)d(y) \\
&= \widetilde{\chi_x}(f)\widetilde{\chi_x}(g)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\widetilde{\chi_x}(f^*) &= \int_{\mathbb{R}^n} \overline{f*(-y)}e^{ix.y}d\lambda(y) \\
&\overset{u=-y}{=} \int_{\mathbb{R}^n} \overline{f(u)e^{ix.u}}d\lambda(u) \\
&= \int_{\mathbb{R}^n} \overline{f(u)e^{ix.u}}d\lambda(u) \\
&= \overline{\widetilde{\chi_x}((f)}
\end{aligned}$$

Hence $\widetilde{\chi_x}$ is a non-trivial homomorphism from $L^1(\mathbb{R}^n)$ to $\mathbb{C}$ hence a character on $L^1(\mathbb{R}^n)$.