Lecture Notes from October 18, 2022

taken by Joseph Walker

0 The Characters of $L^1(\mathbb{R}^n)$

Recall from the previous set of notes that $L^1(\mathbb{R}^n)$ is a Banach-*-Algebra such that for $f \in L^1(\mathbb{R}^n)$ we have $f^*(x) = \overline{f(-x)}$ and $(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(y)d\lambda(y)$.To study the characters of the algebra we considered maps $\mathcal{X}_x : \mathbb{R}^n \to \mathbb{S}^1, y \mapsto e^{ix}y$, which are non-trivial continuous group homomorphisms and thus characters on $\mathbb{R}^n$. The boundedness of these characters inspires the following claim:

0.0.1 Theorem. *For* $f \in L^1(\mathbb{R}^n)$,$\tilde{\mathcal{X}}_x(f) = \int_{\mathbb{R}^n} f(y)e^{ixy}d\lambda(y)$ *defines a character on* $L^1(\mathbb{R}^n)$

Proof. From last time, we have $\tilde{\mathcal{X}}_x(f^*) = \tilde{\mathcal{X}}_x(f^*)$. It remains to show the homomorphism property:

$$\begin{aligned}
\tilde{\mathcal{X}}_x(f * g) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(z-y)d\lambda(y)e^{ixz}d\lambda(z) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)e^{ixz}g(z-y)e^{ix(z-y)}d\lambda(y)d\lambda(z) \\
&= \int_{\mathbb{R}^n} f(y)e^{ixy}\int_{\mathbb{R}^n} g(z-y)e^{ix(z-y)}d\lambda(z)d\lambda(y) \\
&= \tilde{\mathcal{X}}_x(f)\tilde{\mathcal{X}}_x(g)
\end{aligned} \tag{1}$$

□

Thus each x gives a character $\tilde{\mathcal{X}}_x$ on $L^1(\mathbb{R}^n)$. This induces a map from each $f \in \mathbb{R}^n$ to $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ such that $\hat{f}(x) = \tilde{\mathcal{X}}_x(f) = \int_{\mathbb{R}^n} f(y)e^{ixy}d\lambda(y)$. Hence we define $F : L^1(\mathbb{R}^n) \to \mathcal{C}_0(\mathbb{R}^n)$ s.t. $f \mapsto \hat{f}$ and summerize the properties of $\tilde{\mathcal{X}}_{x x\in\mathbb{R}^n}$ as $\hat{f^*} = \overline{\hat{f}}$ and $\widehat{(f * g)} = \hat{f}\hat{g}$.

We now determine how to work with Banach-*-algebras with no unit.

1 Warm-Up

Let A be a Banach Algebra with no unit. Define $\tilde{A} = A \times \mathbb{C} = A \oplus \mathbb{C}$ and identify $(a,\lambda) = (a,0) + (0,\lambda)$. Equip $\tilde{A}$ $(a,\lambda)(b,u) = (ab + \lambda b + ua,\ \lambda u)$ and norm $\|(a, \lambda)\| = \|a\| = |\lambda|$.

Then $\tilde{A}$ is a Banach algebra with unit (0,1) and in fact $A \cong (A, 0)$ is a Banach subalgebra.

We confirm associativity:

$$\begin{aligned} ((a,\lambda)(b,\mu))(c,\delta) &= (ab + \lambda b + \mu a, \lambda\mu)(c,\delta) \\ &= (abc + \lambda bc + \mu ac + \delta ab + \delta\lambda b + \delta\mu\lambda + \lambda\mu c, \lambda\mu\delta) \\ &= (a,\lambda)(bc + \delta b + \mu c, \mu\delta) \\ &= (a,\lambda)((b,\mu)(c,\delta)) \end{aligned} \tag{2}$$

Also (0,1)(a,λ) = (a,λ) for each (a,λ) ∈ $\tilde{A}$ give us the the identity (0,1) for $\tilde{A}$
It remains to show submultiplicativity of the norm: For $(a,\lambda), (b,\mu) \in \tilde{A}$,

$$\begin{aligned} \|(a,\lambda)(b,\mu)\| &= \|(ab + \mu a + \lambda b, \lambda\mu\| \\ &= \|ab + \mu a + \lambda b\| + |\lambda\mu| \\ &\le \|a\|\|b\| + |\mu|\|a\| + |\lambda|\|b\| + |\lambda||\mu| \\ &= (\|a\| + |\lambda|)(\|b\| + |\mu|) \\ &= \|(a,\lambda)\| \cdot \|(b,\mu)\| \end{aligned} \tag{3}$$

Thus (2),(3) and the verification of the identity gives us that is a Banach Algebra with identity.

1.0.1 Remark. $\tilde{A}$ was constructed to deal with Banach Algebras A which have no unit. However, what if A does have a unit ? Then $A \cong (A, 0)$ is a Banach sub-algebra on $\tilde{A}$. However, note that the unit (0,1) $\notin (A, 0)$. So what is the unit of this sub-algebra ? The earlier isomorphism seems to suggest that (1,0), with 1 the identity of A, is the identity of this subalgebra. As it turns out, for each $(a, 0) \in (A, 0)$ we have $(1, 0)(a, 0) = (a, 0)$ and $(1, 0)$ is the identity in the $(A, 0)$ sub-algebra.

2 The Missing Unit

2.0.1 Definition. For a Banach - Algebra A, $a \in A$, we call $\sigma(a) = \{a \in \mathbb{C} : a - \lambda 1 \notin C_0(A)\}$ the spectrum of a and $\rho(a) = \mathbb{C} - \sigma(a)$ the resolvent set. The number $r(a) = inf\{r > 0 : \sigma(a) \subset B_r(0)\}$ is called the spectral radius of a.

Let A be a Banach-Algebra. If A has a unit, then we take $\tilde{A} = A$. Otherwise, we take $\tilde{A}$ as described in the warm up. However, we want $\|1\| = 1$. To achieve this, we have the following theorem:

2.0.2 Theorem. *Let A be a Banach Algebra with unit 1. Then there is a norm $\|\cdot\|_0$ that is equivalent to the norm on A with $\|\cdot\|_0 = 1$, and for $a, b \in A$, we have $\|ab\|_0 \le \|a\|_0 \cdot \|b\|_0$*

Proof. Consider $L_a : A \to A, x \mapsto ax$. Let $\|a\|_0 = \|L_a\| = sup_{\|x\| \le 1}\|ax\| \le \|a\|$. Note that $L_a 1 = a1 = a$. Define $L : A \to B(A) : a \mapsto L_a$. Then $L(x+y) = L_{x+y}$ where for each $z \in A$ we have $L_{x+y}(z) = (x+y)z = xz + yz = L_x z + L_y z$, hence $L(x+y) = L_{x+y} = L_x + L_y = L(x) + L(y)$. Therefore L is linear. Also for each $x, y \in A$ and $a \in A$ we have $L_a(x) = L_a(y)$ implies $ax = ay$ and thus $a(x - y) = 0$ hence $x = y$ (WLOG, let $a = 1$). Thus L_a is one-to-one.

Now we can construct a norm on A with the map $\|a\|_0 = \|L_a\| = \sup_{\|x\|\leq 1} \|ax\| \leq \|a\|$ for each $a \in A$. We show that this is indeed a norm.

(Positive Definiteness) For each $a \in A$, if $\|a\|_0 = 0$, then $\|L_a\| = 0$, hence $L_a = 0$ by positive definiteness of the operator norm. Hence we have $ax = 0$ for any $x \in A$ thus $a = 0$. Also $\|0\|_0 = \|L_0\| = \sup_{\|x\|\leq 1} \|0\| = 0$. Hence $\|a\|_0 = 0$ iff $a = 0$

(Homogeneity) For each $a \in A, \lambda \in \mathbb{C}$, we have $\|\lambda a\|_0 = \|L_{\lambda a}\| = \sup_{\|x\|\leq 1} \|\lambda ax\| = |\lambda| \sup_{\|x\|\leq 1} \|ax\| = |\lambda|\|L_a\| = |\lambda|\|a\|_0$

(Triangle Inequality) For $x, y \in A$ we have $\|x + y\|_0 = \|L(x + y)\| = \|L(x) + L(y)\| \leq \|L_x\| + \|L_y\| = \|x\|_0 + \|y\|_0$

Thus $\|\cdot\|_0$ is positive definite, homogeneous, and satisfies the triangle inequality. Hence $\|\cdot\|_0$ is a norm on A. Also, $\|1\|_0 = \sup_{\|x\|\leq 1} \|1\| = 1$ as needed.

This norm is also sub-multiplicative: For $a, b \in A$, $\|ab\|_0 = \|L_{ab}\| = \|L_a L_b\| \leq \|L_a\|\|L_b\|$

What is left is to show that the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent.

To be continued...

$\square$