Lecture Notes from October 18, 2022

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Last time A Banach $*$-algebra and its relation to the Fourier transform. Recall for $f \in L^1(\mathbb{R}^n)$,

$$\tilde{\chi}_x(f) = \int_{\mathbb{R}^n} f(y)e^{ix \cdot y} d\lambda(y)$$

defines a character. These characters are homomorphisms from the Banach $*$-algebra L^1 to $\mathbb{C}$. We check that it is a homomorphism:

$$\begin{aligned}
\tilde{\chi}_x(f * g) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)g(z-y)d\lambda(y)e^{ix\cdot z}d\lambda(z) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)g(z-y)e^{ix\cdot z}d\lambda(y)d\lambda(z) \\
(Fubini's\ gives) &= \int_{\mathbb{R}^n} f(y)e^{ix\cdot y} \int_{\mathbb{R}^n} g(z)e^{ix\cdot z}d\lambda(z)d\lambda(y) \\
&= \int_{\mathbb{R}^n} f(y)e^{ix\cdot y}\tilde{\chi}_x(g)d\lambda(y)
\end{aligned}$$

Thus, each $x \in \mathbb{R}^n$ defines a character $\tilde{\chi}_x$ on $L^1(\mathbb{R}^n)$. This can be used to map $f \in L^1(\mathbb{R}^n)$ to $\hat{f} : \mathbb{R}^n \to \mathbb{C}$; $\hat{f}(x) = \tilde{\chi}_x(f) = \int_{\mathbb{R}^n} f(y)e^{ix\cdot y}d\lambda(y)$. The properties of $\tilde{\chi}_x(x \in \mathbb{R}^n)$ can be summarized as

$$\widehat{f^*} = \hat{f}^* = \bar{\hat{f}}, \quad (f \widehat{*} g) = \hat{f}\hat{g}.$$

Moreover, $\hat{f} \in C_0(\mathbb{R}^n)$. Define

$$F : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$$
$$f \mapsto \hat{f}$$

F is the Fourier transform on $\mathbb{R}^n$. It is invertible on $range F$ so $L^1(\mathbb{R}^n)$ is isomorphic to a subalgebra of $C_0(\mathbb{R}^n)$. However, it is but not boundedly invertible since it is not onto. If it were onto, then by the bounded inverse theorem the inverse Fourier transform would be continuous (bounded) as a map from C_0 to L^1 and this isn't true. For a counterexample, consider the function $f(x) = \frac{\sin x}{x}e^{-r|x|}$ on $\mathbb{R}$, As $r \to 0$, the Fourier transform converges to the characteristic function of a set while the sup norm stays bounded. However, the L^1-norm of the function goes to infinity which means that the inverse is unbounded.

We will also see later that the set $\{\tilde{\chi}_x\}_{x\in\mathbb{R}^n}$ exhausts all characters.

Warm up: (*Mystery of the missing unit*)

1.47 Question. What if a Banach $*$-algebra does not have a unit?

Let $\mathcal{A}$ be a Banach algebra and let $\tilde{\mathcal{A}} = \mathcal{A} \times \mathbb{C}$. Identify this Cartesian product as $\mathcal{A} \times \mathbb{C}$, $(a, \lambda) = (a, 0) + (0, \lambda)$ equipped with multiplication $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$, and norm $\|(a, \lambda)\| = \|a\| + |\lambda|$. Then $\tilde{\mathcal{A}}$ is a Banach algebra with unit $(0, 1)$ and $\mathcal{A}$ embeds in $\tilde{\mathcal{A}}$, $\mathcal{A} \cong (\mathcal{A}, 0) \leq \tilde{\mathcal{A}}$ as a subalgebra.

Note that if $\mathcal{A}$ is $*$-algebra also then $(a, \lambda)^* = (a^*, \bar{\lambda})$ defines an involution on $\mathcal{A}$.

We confirm associativity:

$$\begin{aligned}
((a, \lambda)(b, \mu))(c, \gamma) &= (ab + \lambda b + \mu a, \lambda\mu)(c, \gamma) \\
&= (abc + \lambda bc + \mu ac + \gamma ab + \lambda\mu c + \gamma\lambda b + \gamma\mu a + \lambda\mu\gamma) \\
&= (a, \lambda)(bc + \mu c + \gamma b, \gamma\mu) \\
&= (a, \lambda)((b, \mu))(c, \gamma)).
\end{aligned}$$

Also, for all $(a, \lambda) \in \tilde{\mathcal{A}}$,

$$(0, 1)(a, \lambda) = (a, \lambda) = (a, \lambda)(0, 1).$$

We check sub-multiplicativity next, for all $(a, \lambda), (b, \mu) \in \tilde{\mathcal{A}}$,

$$\begin{aligned}
\|(a, \lambda)(b, \mu)\| &= \|(ab + \lambda b + \mu a, \lambda\mu)\| \\
&= \|ab + \lambda b + \mu a\| + |\lambda\mu)| \\
&\leq \|ab\| + |\lambda|\|b\| + |\mu|\|a\| + |\lambda\mu| \\
&\leq \|a\|\|b\| + |\lambda|\|b\| + |\mu|\|a\| + |\lambda\mu| \\
&\leq \|a\|(\|b\| + |\mu|) + |\lambda(\|b\| + |\mu|) \\
&\leq (\|a\| + |\lambda|)(\|b\| + |\mu|) \\
&\leq \|(a, \lambda)\|\|(b, \mu)\|.
\end{aligned}$$

Finally, we check completeness: let (a_n, λ_n) be a Cauchy sequence in $\tilde{\mathcal{A}}$, then there existas $N \in \mathbb{N}$ such that $\|(a_n, \lambda_n) - (a_m, \lambda_m\| \to 0$ for all $n, m > N$. Using the norm defined, we get that the sequences $\|a_n - a_m\| \to 0$ and $|\lambda_n - \lambda_m| \to 0$ for all $n, m > N$ giving us Cauchy sequences $(a_n) \subset \mathcal{A}$ and $(\lambda_n) \subset \mathbb{C}$. By the completeness of $\mathcal{A}$ and $\mathbb{C}$, we have $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ such that $(a_n) \to a$ and $(\lambda_n) \to \lambda$. Again, using the norm defined, we have that $\|a_n - a\| + |\lambda_n - \lambda| \to 0$ and thus $(a_n, \lambda_n) \to (a, \lambda)$ in $\tilde{\mathcal{A}}$. Using these properties we can proceed with Banach algebras even without a unit.

However, let us consider the case when $\mathcal{A}$ has a unit, $1_{\mathcal{A}}$, and we embed it into $\tilde{\mathcal{A}}$. Due to uniqueness of the unit, $(0, 1)$ is the only unit in $\tilde{\mathcal{A}}$. Consider the element $(1_{\mathcal{A}}, 0)$ in $\tilde{\mathcal{A}}$, for any $(a, \lambda) \in \tilde{\mathcal{A}}$,

$$(1_{\mathcal{A}}, 0)(a, \lambda) = (a + \lambda, 0)$$

and so we see that multuplication by $(1_{\mathcal{A}}, 0)$ gives us a projection onto $\mathcal{A}$.

The missing unit: If $\mathcal{A}$ has a unit, we take $\tilde{\mathcal{A}} = \mathcal{A}$; if not we take $\tilde{\mathcal{A}}$ as above.

1.48 Definition (Spectrum)**.** For a Banach algebra $\mathcal{A}$, $a \in \mathcal{A}$ we call

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin G(\tilde{\mathcal{A}})\}$$

the spectrum of a and $\rho(a) = \mathbb{C} \setminus (\sigma(a))$ the resolvent of a.

1.49 Definition (Spectral radius)**.** The number $r(a) = \inf\{r > 0 : \sigma(a) \subset B_r(0)\}$ is called the spectral radius of a.

Note: If $\mathcal{A}$ does not have a unit and we adjoin 1 to it, then $\|1\| = 1$. We want to achieve this if $\mathcal{A}$ has a unit, if necessary by moving to an equivalent norm.

1.50 Theorem. *Let $\mathcal{A}$ be a Banach algebra with unit 1, then there exists a norm $\|\cdot\|_0$ that is equivalent to $\|\cdot\|$ (the norm on $\mathcal{A}$) and $\|1\|_0 = 1$, and for $a, b \in \mathcal{A}$, $\|ab\|_0 \leq \|a\|_0\|b\|_0$.*

Proof. Consider for $a \in \mathcal{A}$, the map $L_a : \mathcal{A} \to \mathcal{A}$, $x \mapsto ax$ and let

$$\|a\|_0 := \|L_a\| = \sup_{\|x\|\leq 1} ax \leq \|a\|.$$

The map

$$L : \mathcal{A} \to \mathbb{B}(\mathcal{A})$$
$$a \mapsto L_a$$

is linear: $\forall x$, $L(a+b)(x) = L_{a+b}(x) = (a+b)(x) = ax + bx = L_a x + L_b x \implies L(a+b) = L_a + L_b$. Also, it is one-one: since $L_a.1 = a$, if $L_a = L_b$ then $\forall x,\ ax = bx$ and for $x = 1$ we get $a = b$. Thus the norm $\|\cdot\|_0$ norm given by $\|a\|_0 := \|L_a\|$ is a well-defined norm on $\mathcal{A}$. Also,

$$\|1\|_0 = \|1\| = \|id_{\mathcal{A}}\| = \sup_{\|x\|\leq 1} x = 1$$

and note that for all x, $L_{ab}(x) = (ab)x = a(bx) = L_a L_b(x)$ hence $L_{ab} = L_a L_b$. Thus

$$\begin{aligned} \|ab\|_0 &= \|L_{ab}\| \\ &= \|L_a L_b\| \\ &\leq \|L_a\|\|L_b\| \end{aligned}$$

since the norm on $\|L_a\|$ is the operator norm from $\mathbb{B}(\mathcal{A})$ that is sub-multiplicative. Hence $\|ab\|_0 \leq \|a\|_0\|b\|_0$. Next class, we show the equivalence of these norms. □