# MATH 7320 Lecture Notes

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**Last time:**

- Banach Algebra with and without unit.
- We had proved most of the following.

**Theorem 1.** *Let $A$ be a Banach algebra with unit 1, then there exist a norm $\|.\|_0$ that is equivalent to on $A$ and $\|1\|_0 = 1$, and for $a, b \in A$*

$$\|ab\|_0 \leq \|a\|_0 \|b\|_0$$

.

**Proof:** Consider $L_a : A \longrightarrow A$ defined by $x \longrightarrow ax$.

$$\|a\|_0 = \|L_0\| = \sup_{\|x\| \leq 1} \|ax\| \leq \|a\| \ .$$

Because of $L_a 1 = a$, the linear map $L : A \longrightarrow \mathcal{B}(A)$, $a \longrightarrow L_a$ is one-one. First we prove the map $L$ is linear as follows
Let $\alpha,\ \beta \in \mathbb{C}$ and $a,\ b \in A$ then

$$L(\alpha a + \beta b) = L_{(\alpha a + \beta b)} \ . \tag{0.1}$$

Then for all $x \in A$, we have

$$\begin{aligned} L_{(\alpha a + \beta b)}(X) &= (\alpha a + \beta b)x \\ &= \alpha a x + \beta b x = \alpha(ax) + \beta(bx) \\ &= (\alpha L_a + \beta L_b)x \end{aligned}$$

Thus, by equation (0.1)

$$L(\alpha a + \beta b) = L_{(\alpha a+\beta b)} = \alpha L_a + \beta L_b$$

This implies that $L$ is linear. Now by norm properties of $\|.\|$, we see that $\|.\|_0$ is a norm on $A$.

$$\|a\|_0 = 0 \iff \|L_a\| = 0 \iff L_a = 0 \iff a = 0 \ ,$$

as $L$ is one-to-one. Also, $\|\alpha a\|_0 = \|L_{\alpha a}\| = \|\alpha L_a\| = |\alpha|\|L_a\| = |\alpha|\|a\|_0$ for all $\alpha \in \mathbb{C}$ and by linearity of $L$, we have

$$\|a+b\|_0 = \|L_{a+b}\| = \|L_a + L_b\| \leq \|L_a\| + \|L_b\| = \|a\|_0 + \|b\|_0 \ ,$$

$$\implies \quad \|a+b\|_0 \leq \|a\|_0 + \|b\|_0$$

So, $\|a\|_0 = \|L_a\|$ is a norm on $A$.
Furthermore, $\|1\|_0 = \sup_{\|x\|\leq 1} \|1x\| = 1$, and

$$\|ab\|_0 = \|L_a\| = \|L_a L_b\| \leq \|L_a\|\|L_b\| = \|a\|_0\|b\|_0 \ .$$

Now it is left to show that $\|.\|$ and $\|.\|_0$ are equivalent. To see this

$$\|a\| = \|a_1\| = \|L_a 1\| \leq \|L_a\|\|1\| = \|a\|_0\|1\| \leq \|a\|\|1\| \ .$$

So,

$$\frac{1}{\|1\|}\|a\| \leq \|a\|_0 \leq \|a\| \ .$$

Here , the equivalence of norms implies that the algebra with the new norms is also a Banach algebra. From now on, we assume that if 1 is a unit in a Banach algebra, then we can assume $\|1\| = 1$. Next we study $C^*-$algebras, where $\|a\| = \|L_a\|$ for each $a \in A$.

# 1 Properties of the embedding $A \longrightarrow \tilde{A}$ when $A$ is a $C^*-$algebra

**Theorem 2.** *Let $A$ be a $C^*-$ algebra, then*

1. *If $L_a : x \longrightarrow ax$ as above, then $\|a\| = \|L_a\|$. In particular, if 1 is a unit, then $\|1\| = 1$.*

*2. If $A$ does not have a unit, then $\tilde{A}$ becomes a $C^*-$algebra if we define*

$$(a,\lambda)^* = (a^*, \overline{\lambda})$$

*and we choose the norm $\|(a,\lambda)\| = \|L_{(a,\lambda)}\|$, where for $x \in A$,*

$$1_{(a,\lambda)}x = ax + \lambda x \ .$$

**Proof:** 1. We have

$$\|L_a\| = \sup_{\|x\|\leq 1} \|L_a x\| = \sup_{\|x\|\leq 1} \|ax\| \leq \|a\| \ .$$

On the other hand

$$\|aa^*\| = \|a\|^2 = \|a\|\|a^*\|$$

So,

$$\|L_a\| = \sup_{\|x\|\leq 1} \|L_a x\| \geq \|L_a \frac{a^*}{\|a\|}\| = \|a\frac{a^*}{\|a\|}\| = \|a\| \ .$$

We conclude, $\|a\| = \|L_a\|$. From 1 being unit $L_1 = id_A$,

$$\|1\| = \|L_1\| = 1 \ .$$

2. $L : \tilde{A} \longrightarrow \mathcal{B}(A)$ be given by

$$L(a,\lambda) = L_{(a,\lambda)} \ \ \& \ \ L_{(a,\lambda)}x = ax + \lambda x \ .$$

We show $L$ is one-one. Let $L(a,\lambda) = 0$. If $\lambda = 0$, then $L_a = 0$ and so $a = 0$. If $\lambda \neq 0$, then by linearity

$$\begin{aligned} L_{(a,\lambda)}x &= 0, \\ ax + \lambda x &= 0 \\ \Longrightarrow \quad \left(\frac{-1}{\lambda}\right) ax - x &= 0 \end{aligned}$$

$\Longrightarrow \quad \left(\frac{-1}{\lambda}\right) a$ is (left) unit. This contradicts our assumption that $A$ does not have a unit. Thus, $L$ is one-one and $\|(a,\lambda)\| = \|L_{(a,\lambda)}\|$ is a norm on $A$.
To check the norm property, we only need to show for $x \in \tilde{A}$ ,

$$\|x\|^2 \leq \|x^*x\| \ ,$$

where the norm is defined by $\|x\| = L_x$. If $\|x\| = 0$, then there is nothing to show. If $0 < r < \|x\|$, by definition of norm on $\tilde{A}$, there is $y \in A$ , $\|y\| \leq$

1 , $\|xy\| \geq r$. For $x \in \tilde{A}$ and $y \in A$, by the multiplication law $xy \in A$. Replacing $x$ with $xy$ gives,

$$\begin{aligned} \|x^*x\| &= \|y^*x^*xy\| \\ &= \|(xy)^*xy\| \\ &= \|xy\|^2 \geq r^2 \ . \end{aligned}$$

So taking the supremum over $r < \|x\|$ gives $\|x^*x\| \geq \|x\|^2$. We conclude with examples.

**Example 3.** *Let $X$ be a locally compact Hausdorff space that is not compact. Let $A = C(X)$, then $A$ does not have a unit (why?) and $\tilde{A}$ can be thought of as continuous functions with a limit at infinity, with $(0,1) \equiv 1$.*

To justify this, we note $\tilde{A} \longrightarrow C_b(X)$, $(f, \lambda) = f + \lambda 1$ can be thought of as an isometry, where $C_b(X)$ has the supremum.
We want $(f, \lambda) = f + \lambda 1$. If we choose $\|(f, \lambda)\| = \|f\|_\infty + |\lambda|$ , then $\|f + \lambda 1\|_\infty \leq \|f\|_\infty + |\lambda|$. But equality may not hold. If instead we let $\|(f, \lambda)\| = \|L_{(f,\lambda)}\|$, then Urysohn's Lemma shows

$$\|L(f, \lambda)\| = \|f + \lambda 1\|_\infty \ .$$

Hence, we can think of $C_0\tilde{(}X)$ as functions that have a limit at $\infty$, equipped with Sup. norm .

## 2 Examples of $C^*-$algebra and spectra of elements

**Example 4.** *Let $X$ be a compact Hausdorff space, $A = C(X)$, $f \in A$. What is $\sigma(f)$ ?*

We recall $g \in G(A)$ means there exists $h \in C(X)$ and $gh = 1$. So $g(x) \neq 0$ for each $x \in X$.
Conversely, if $g(x) \neq 0$ for each $x \in X$, then $h(x) = \frac{1}{g(x)}$ is in $C(X)$.
Next, to see what the spectrum is $f - \lambda 1$ is invertible if and only if $f(x) \neq \lambda$ at any $x \in X$. Consequently , $\sigma(f) = f(X)$.

**Example 5.** *Let $X$ be a locally compact Hausdorff space, $A = C_0(X)$, $f \in A$. What is $\sigma(f)$?*

In order to invert a function in a bounded manner, assuming $f \in C(X)$ and $f$ has limit at $\infty$, then $f(X) \neq 0$ for each $x$, and $\lim_{x \longrightarrow \infty} f(x) \neq 0$. In notation of $\tilde{A}$, we need to embed $A$ in $\tilde{A}$ by $f \longrightarrow (f, \lambda)$, where $(f, \lambda) = f + \lambda 1$ and $f(x)$ has a limit at $\infty$. So, $(f, \lambda)$ is invertible if and only if $f(x) \neq 0$, for each $x \in X$ and $\lambda \neq 0$. Hence, $\sigma(f) = f(X) \cup \{0\}$.

**Example 6.** *Let $A \subset \mathcal{B}(\mathbb{C}^n)$ be an algebra of $n \times n$ matrices containing $1$. Let $x \in A$ is invertible in $A$ if and only if there is $y \in A$ such that $yx = xy = 1$.*

We show that if there is $y \in \mathcal{B}(\mathbb{C}^n)$ with $xy = yx = 1$, then $y \in A$. To see this, note $L_x : A \longrightarrow A$. So, if $x$ is invertible in $\mathcal{B}(\mathbb{C}^n)$, then $y \longrightarrow xy$ is one-to-one. Since, $A$ is finite dimensional, $L_x$ is also onto. So, there is a $yA$ such that

$$L_x(y) = xy = 1 \ .$$

Hence, $y = x^{-1} \in A$. Thus ,

$$\begin{aligned} G(A) &= A \cap G(\mathcal{B}(\mathbb{C}^n)) \\ &= \{a \in A : \ det(a) \neq 0\} \end{aligned}$$

From this, we deduce for $a \in A$

$$\sigma(a) = \{\lambda \in \mathbb{C} : \ a - \lambda 1 \notin G(A)\} \ .$$

Hence, the spectrum of $a$ consists of the eigenvalues of $a$. It is interesting to note, $\sigma(a)$ does not depend on the choice of $A \subset \mathcal{B}(\mathbb{C}^n)$.