Lecture Notes from October 20, 2022

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Last time

- Banach algebra with and without a unit
- We had proved most of the following:

2.2 Theorem. *Let A be a Banach algebra with unit $\mathbb{1}$, then there exists a norm $\|.\|_0$ that is equivalent to the norm on A and satisfies $\|\mathbb{1}\|_0 = 1$, and for each $a, b \in A$*

$$\|ab\|_0 \leq \|a\|_0 \|b\|_0$$

Proof. We had $\|a\|_0 = \|L_a\| = \sup_{\|x\|\leq 1} \|ax\| \leq \|a\|$. It remains to show that $\|.\|$ and $\|.\|_0$ are equivalent. To see this, note

$$\|a\| = \underbrace{\|a\mathbb{1}\|}_{L_a \cdot (\mathbb{1})} \leq \|L_a\| \|\mathbb{1}\| = \|a\|_0 \|\mathbb{1}\| \leq \|a\| \, \|\mathbb{1}\|$$

so

$$\frac{1}{\|\mathbb{1}\|} \|a\| \leq \|a\|_0 \leq \|a\|$$

□

From now on, we assume that if $\mathbb{1}$ is a unit in a Banach algebra, then we can assume $\|\mathbb{1}\| = 1$. Next, we study C^*-algebras where $\|a\| = \|L_a\|$ for each $a \in A$.

2.3 Theorem. *Let A be a C^*-algebra, then*

(1) If $L_a : x \mapsto ax$ as above, then $\|a\| = \|L_a\|$. In particular, if $\mathbb{1}$ is a unit, then $\|\mathbb{1}\| = 1$

(2) If A does not have a unit, then $\tilde{A}$ becomes a C^*-algebra if we define $(a, \lambda)^* = (a^*, \bar{\lambda})$, and we choose the norm $\|(a, \lambda)\| := \|L_{(a,\lambda)}\|$ for $x \in A$

$$L_{(a,\lambda)} x = ax + \lambda x$$

Proof. (1) We have

$$\|L_a\| = \sup_{\|x\|\leq 1} \|L_a x\| = \sup_{\|x\|\leq 1} \|ax\| \leq \|a\|$$

On the other hand, $\|a\,a^*\| = \|a\|^2 = \|a\|\,\|a^*\|$, so if $a = 0$, nothing to show.
Suppose $a \neq 0$, we let $x = \frac{a^*}{\|a\|}$ and consider

$$\|L_a\| = \sup_{\|x\|\leq 1} \|L_a x\| \geq \|L_a \frac{a^*}{\|a\|}\|$$
$$= \|a \frac{a*}{\|a\|}\| = \|a\|$$

We conclude $\|a\| = \|L_a\|$. From $\mathbb{1}$ being a unit, $L_{\mathbb{1}} = id_A$, and $\|\mathbb{1}\| = \|L_{\mathbb{1}}\| = 1$

(2) Let $L : \tilde{A} \mapsto \mathcal{B}(A)$ given by $L(a,\lambda) = L_{(a,\lambda)},\ L_{(a\lambda)}x = ax + \lambda x$. We omitted the proof that $\tilde{A}$ is a Banach space. We will only show the norm property of a C^*-algebra.

First, we show L is 1-1. Let $L(a,\lambda) = 0$. If $\lambda = 0$, then $L_a = 0$ so a=0. Suppose $\lambda \neq 0$, then by linearity,

$$0 = L_{(a,\lambda)}x = ax + \lambda x \implies (-\frac{1}{\lambda})ax - x = 0$$

implies that $(-\frac{1}{\lambda})a$ is a (left) unit in A which contradicts our assumption that A does not have a unit. Thus L is 1-1, and $\|(a,\lambda)\| = \|L(a,\lambda)\|$ is a norm which extends the norm on A.

To check the norm property, we only need to show for $x \in \tilde{A},\ \|x\|^2 \leq \|x^*\,x\|$.
If $\|x\| = 0$, nothing to show.
If $0 < r < \|x\|$, by definition of norm on $\tilde{A}$ and the result above, we have

$$r < \|x\| = \|L_x\| = \sup_{\|y\|\leq 1} \|L_x y\| = \sup_{\|y\|\leq 1} \|xy\|$$

so there is $y \in A,\ \|L_x y\| = \|xy\| \geq r$.
Using the submultiplicity property of $\tilde{A}$, consider $y \equiv (y, 0)$, we have

$$\|x^*\,x\| \overset{\|y\|=\|y^*\|\leq 1}{\geq} \|y^*\|\,\|x^*\,x\|\,\|y\| \overset{\text{submult}}{\geq} \|y^* x^*\,xy\| = \|(xy)^* xy\| \overset{xy\in A:\,C^*\text{-algebra}}{=} \|xy\|^2 \geq r^2$$

so taking the sup over all $r < \|x\|$ gives $\|x^*\,x\| \geq \|x^2\|$ ☐

We conclude with examples

2.4 Example. Let X be a locally compact Haudorff space that is not compact, let $A = C_0(X)$, then A does not have a unit and $\tilde{A}$ can be thought of continuous functions with limit at infinity, with $(0,1) \equiv \mathbb{1}$. To justify this, we note $\tilde{A} \to C_b(X),\ (f,\lambda) \mapsto f + \lambda\mathbb{1}$ can be thought if as an isometry, where $C_b(X)$ has the sup-norm.

2.5 Remark. $C_0(X)$ does not have a unit since the constant function $\mathbb{1}$ is not included in A as it does not go to 0 at infinity.

Proof. We want to identify $(f,\lambda) = f + \lambda\mathbb{1}$.
If we choose $\|(f,\lambda)\| = \|f\|_\infty + |\lambda|$, then $\|f + \lambda\mathbb{1}\|_\infty \leq \|f\|_\infty + |\lambda|$ but the equality may not hold. If instead, we choose $\|(f,\lambda)\| = \|L_{(f,\lambda)}\|$, then Urysohn's lemma guarantees the existence of a function $g = (f,\lambda) \in \tilde{A}$ such that $g(x) = 1$ for $x \in K$=compact, and $g(x) = 0$ where $x \notin K$.

$$\|L_{(f,\lambda)}\| = \sup_{\|x\|\leq 1} |f(x) + \lambda x| = \|f + \lambda\mathbb{1}\|_\infty$$

Hence $\widetilde{C_0(X)}$ is a closed subspace of $C_b(X)$ which is isometrically embedded in $C_b(X)$ as a space of continuous functions that have limit at infinity. □

2.6 Example. Let X be compact Hausdorff space $A = C(X)$, $f \in A$. What is $\sigma(f)$?

We recall $g \in \mathcal{G}(A)$ means there exists $h \in C(X)$ and $gh = \mathbb{1}$ so $g(x) \neq 0$ for each $x \in X$. Conversely, if $g(x) \neq 0$ for each $x \in X$, then $h = \frac{1}{g(x)}$ is in $C(X)$.
Next, to see what the spectrum is, note $f - \lambda\mathbb{1}$ is invertible **if and only if** $f(x) \neq \lambda$ at any $x \in X$. Consequently, $\sigma(f) = \{\lambda : f(x) = \lambda \text{ for some } x \in X\} = f(X)$.

2.7 Example. Let X be a locally compact Hausdorff space $A = C_0(X)$. What is $\sigma(f)$?

For $f \in \tilde{A}$, then $f \in C(X)$ and f has limit at infinity. So if f is invertible, then $f(x) \neq 0$ for each x, and $\lim_{x\to\infty} f(x) \neq 0$. Otherwise, taking $1/f$ would diverges at infinity, so not give a function in $\tilde{A}$. In notation of $\tilde{A}$, (f,λ) is invertible iff $f(x) \neq 0$ for each x and $\lambda \neq 0$. Hence $\sigma(f) = f(X) \cup \{0\}$.

2.8 Example. Let $A \subset B(\mathbb{C}^n)$ be an algebra of $n \times n$ matrices containing $\mathbb{1}$. For $a \in A$, what is $\sigma(a)$?

Let $x \in A$ be invertible in A **if and only if** there is $y \in A$ s.t $xy = yx = \mathbb{1}$. We show that if there is $y \in B(\mathbb{C}^n)$ with $xy = yx = \mathbb{1}$ then $y^{-1} \in A$. To see this, note $L_x : A \to A$, so if x is invertible in $B(\mathbb{C}^n)$, then the map $y \mapsto xy$ is 1-1. Since A is finite dimensional, L_x is also onto. So there exists $y \in A$ s.t $L_x(y) = xy = \mathbb{1}$. Hence $x^{-1} = y \in A$.
Thus $\mathcal{G}(A) = A \cap \mathcal{G}(B(\mathbb{C}^n)) = \{a \in A : \det a \neq 0\}$.
From this, we deduce for $a \in A$, $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda\mathbb{1} \notin \mathcal{G}(A)\}$. Hence the spectrum consists of eigenvalues of a.

2.9 Remark. It is interesting to note $\sigma(a)$ does not depend on the choice of $A \subset B(\mathbb{C}^n)$