The tale of two norms on $\tilde{A}$ and Properties of the spectrum

Lecture Notes from October 25, 2022
taken by An Vu

Last Time

- Properties of the embedding $A \to \tilde{A}$ when A is a a C*-algebra.
- Examples of C*-algebra and spectra of elements.

Recap: A Tale of two norms on $\tilde{A}$

1. Let A be a Banach algebra, $\tilde{A} = A \times \mathbb{C} = A \oplus \mathbb{C}$, with norm $\|(a, \lambda)\| = \|a\| + |\lambda|$. Then $\tilde{A}$ is Banach. This has been shown in the Warm-up in October 18 notes.

   When considering $A = C_0(X)$, X locally compact but not compact, if for $f \in A$, $f : X \to \mathbb{C}$, $f(x) \geq 0$ for all $x \in X$, and $\|f\|_\infty = 1$, then

   $$\|(f, 1)\| = \|f\|_\infty + 1 = 2,$$

   and

   $$\|(-f, 1)\| = \| - f\|_\infty + 1 = 2,$$

   so this does not coincide with

   $$\|1 + f\|_\infty = 2,$$

   and

   $$\|1 - f\|_\infty = 1.$$

   Rudin commented that this norm is a good start, but it does not work quite the way we would like it to be obtained from a norm on a function space, e.g. the sup-norm.

2. We consider another norm on $\tilde{A}$ induced by L,

   $$\|(a, \lambda)\| = \|L_{(a,\lambda)}\| = \sup_{\substack{x \in A \\ \|x\| \leq 1}} \|ax + \lambda x\|.$$

When considering $A = C_0(X)$,

$$\|(a,\lambda)\| = \sup_{\substack{f\in C_0(X)\\ \|f\|_\infty\le 1\\ x\in X}} \|a(x)f(x) + \lambda f(x)\|,$$

and indeed, $\|(a,\lambda)\| = \|a + \lambda 1\|_\infty$.

In case (1), $\tilde{A}$ has been shown to be Banach, but in case (2), to show $\tilde{A}$ is Banach, i.e. complete, we need to recall the following Lemma.

Warm up

0.0 Lemma. *Let ϕ be a linear functional on a normed space. Then ϕ is bounded if and only if $\ker\phi$ is closed.*

Proof. Assume ϕ is bounded. Then by continuity, $\ker\phi$ is closed. Next, we prove the converse by contrapositive, i.e. we need to show if ϕ is not bounded, then $\ker\phi$ is not closed:

If ϕ is unbounded, then there is a sequence $(x_n) \in A$ such that for each $n \in \mathbb{N}, \|x_n\| \le 1$ and $|\phi(x_n)| \to \infty$.

Consider $a \notin \ker\phi$, i.e. $\phi(a) \neq 0$, and choose

$$y_n = a - \frac{x_n}{\phi(x_n)}\phi(a).$$

Note that $\phi(x_n) = 0$ for some n is not a problem. We see that $\phi(y_n) = \phi(a) - \frac{\phi(x_n)}{\phi(x_n)}\phi(a) = 0$, so each y_n is in $\ker\phi$. We also see $y_n \to a$ by $\phi(x_n) \to \infty$, but $a \notin \ker\phi$. Thus, $\ker\phi$ is not closed. □

We are now ready to complete the material from last time.

1 The Banach space $\tilde{A}$

1.1 Proposition. *Let A be a C*-algebra without unit, then $\tilde{A}$, equipped with the norm induced by L, is a Banach space.*

Proof. For $(a,\lambda) \in \tilde{A}$, let $\pi_2(a,\lambda) = (0,\lambda)$ be a linear functional, then

$$\ker\pi_2 = (A,0) \cong A,$$

since $\|(a,0)\| = \|a\| + 0 = \|a\|$ for each $a \in A$, and by completeness of A, $\ker\pi_2$ is closed.

Then, by previous Lemma, π_2 is a bounded linear map.

Consequently, $\pi_1(a,\lambda) := (a,0)$ is bounded because

$$\pi_1(a,\lambda) = (a,\lambda) - \pi_2(a,\lambda).$$

Take a Cauchy sequence $(a_n,\lambda_n) \in \tilde{A}$, then $\pi_1(a_n,\lambda_n)$ is Cauchy, and so is $\pi_2(a_n,\lambda_n)$, and by completeness of $A\times\{0\}$ and $\{0\}\times\mathbb{C}$, we have $a_n \to a$ and $\lambda_n \to \lambda$.

Next, by

$$\|L_{(b,\mu)}\| = \|L_{(b,0)} + L_{(0,\mu)}\| \leq \|L_{(b,0)}\| + \|L_{(0,\mu)}\|,$$

we get

$$\|L_{(a_n,\lambda_n)} - L_{(a,\lambda)}\| = \|L_{(a_n,0)} + L_{(0,\lambda_n)} - L_{(a,0)} - L_{(0,\lambda}\| \leq \|L_{(a_n,0)} - L_{(a,0)}\| + \|L_{(0,\lambda_n)} - L_{(0,\lambda}\|,$$

and since $a_n \to a$ and $\lambda_n \to \lambda$, $L_{(a_n,\lambda_n)} - L_{(a,\lambda)} \to 0$ and $L_{(0,\lambda_n)} - L_{(0,\lambda)} \to 0$ by A being a C*-algebra. Thus, $\|L_{(a_n,\lambda_n)} - L_{(a,\lambda)}\| \to 0$, and thanks to completeness of $A \times \{0\}$ and $\{0\} \times \mathbb{C}$ again, we get that $L_{(a_n,\lambda_n)} \to L_{(a,\lambda)}$. Therefore, $\tilde{A}$ equipped with the L-induced norm is complete, and hence, Banach. □

2 Properties of the Spectrum

In the case of finite dimensional complex Hilbert spaces, we saw the spectrum is non-empty because the characteristic polynomial of a matrix has at least one root (by Fundamental Theorem of Algebra).

In this section, we show that, for $a \in A$, where A is a Banach algebra, $\sigma(a)$ is non-empty, but first, we need some complex analysis.

2.1 Theorem. *Suppose A is a Banach algebra with unit 1, and $\|1\| = 1$. We have the following properties:*

1. *For $\|x\| < 1$, $1 - x$ is invertible, and $(1-x)^{-1} = \sum_{n=0}^{\infty}$. This series is called Neumann series. Moreover,*
$$\|(1-x)^{-1}\| = \frac{1}{1 - \|x\|},$$
and
$$\|(1-x)^{-1} - 1\| \leq \frac{\|x\|}{1 - \|x\|}.$$

2. *$G(A)$ is an open subset of A. More precisely, if $r = \frac{1}{\|a^{-1}\|}, B_r(a) \subset G(A)$, for each $a \in G(A)$. Also, $G(A)$ forms a group whose operations (multiplication and inverse) are continuous.*

3. *For each $a \in A$, $\sigma(a)$ is a compact subset of $\mathbb{C}$, and $r(a) \leq \|a\|$, where $r(a)$ is called the spectral radius of a.*

4. *The function R from the resolvent set $\rho(a)$ of a to A, defined by $\lambda \mapsto (a - \lambda 1)^{-1}$, is continuous and even analytic, i.e. for each $a \in A, \lambda_0 \in \rho(a)$, there is $r > 0$ such that, for $\lambda \in B_r(\lambda_0)$,*
$$R(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda_0)(\lambda - \lambda_0)^n,$$
with some sequence $(a_n(\lambda_0)) \in A$ such that the series $R(\lambda)$ converges in the norm of A.

Proof. 1. The convergence of the Neumann series follows from

$$\sum_{n=0}^{\infty} \|x^n\| \leq \sum_{n=0}^{\infty} = \frac{1}{1 - \|x\|} \leq \infty.$$

Let $y = \sum_{n=0}^{\infty} x^n$. Then

$$y(1-x) = (1-x)y = (1 + x + x^2 + x^3 + ...) - (x + x^2 + x^3 + ...) = 1,$$

so $y = (1-x)^{-1}$.

The norm of y, by the above definition of y, is bounded by $\|y\| \leq \frac{1}{1-\|x\|}$.

The second norm estimate follows from

$$\|y - 1\| = \|\sum_{n=1}^{\infty} x^n\| \leq \|x\| \sum_{n=0}^{\infty} \|x^n\| = \|x\| \frac{1}{1 - \|x\|}.$$

2. From submultiplicativity of A, we get that the map $(a, b) \mapsto ab$ is continuous on $A \times A$. From part (1), we also see that $z \mapsto z^{-1}$ is continuous at 1, since if $x_n \to 0$, then $(1 - x_n)^{-1} \to 1$. The rest of the proof will be in October 27 notes.

□