Lecture Notes from October 25, 2022

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Last time

- Properties of the embedding $A \mapsto \tilde{A}$ when A is a C*-algebra
- Examples of C*-algebras and spectra of elements

Recap: A tale of two norms on $\tilde{A}$

i) Let A be a Banach algebra, $\tilde{A} = A \times \mathbb{C} = A \oplus \mathbb{C}$ with norm $\|(a, \lambda)\| = \|a\| + |\lambda|$ for each $(a, \lambda) \in \tilde{A}$, then $\tilde{A}$ is Banach. As an example, let us consider $A = C_0(X)$, X locally compact Hausdorff but not compact. If $f : X \mapsto \mathbb{C}$, $f(x) \geq 0$ for each $x \in X$, $\|f\|_\infty = 1$, then

$$\|(f, 1)\| = \|f\|_\infty + 1 = 2$$

and

$$\|(-f, 1)\| = \|-f\|_\infty + 1 = 2$$

which does not coincide with

$$\|1 + f\|_\infty = 2$$

and

$$\|1 - f\|_\infty = 1.$$

This illustrates that we might want to consider another norm.

ii) Because we would like to embed $\tilde{A}$ obtained from $A = C_0(X)$ in a function space with the sup-norm, we consider another norm on $\tilde{A}$ induced by a linear operator L, namely,

$$\begin{aligned}\|(a, \lambda)\| &= \|L_{(a,\lambda)}\| \\ &= \sup_{\substack{x \in A \\ \|x\| \leq 1}} \|ax + \lambda x\|\end{aligned}$$

Now, if we again consider the example where $A = C_0(X)$, but use the norm we just introduced, i.e.

$$\|(a, \lambda)\| = \sup_{\substack{f \in C_0(X) \\ \|f\|_\infty \leq 1 \\ x \in X}} |a(x)f(x) + \lambda f(x)|$$

then indeed $\|(a, \lambda)\| = \|a + \lambda 1\|_\infty$. To see this, first note that we have $\|(a, \lambda)\| \leq \|a + \lambda 1\|_\infty$. Then, to show that $\|(a, \lambda)\| \geq \|a + \lambda 1\|_\infty$, the idea is to use Urysohn's Lemma for locally compact hausdorff spaces to construct a specific function $f \in C_0(X)$ with $\|f\|_\infty \leq 1$ that will establish the lower bound.

iii) Problem: Is $\tilde{A}$ with this norm complete?

Warm up:

1.6 Lemma. *Let Φ be a linear functional on a normed space. Then, Φ is bounded if and only if $\Phi^{-1}(\{0\})$ is closed.*

Proof. If Φ is bounded, then by continuity $\Phi^{-1}(\{0\})$ is closed. To prove the converse, we will show that if Φ is not bounded then $\Phi^{-1}(\{0\})$ is not closed. Suppose that Φ is unbounded, then there exists a sequence (x_n) in A such that for each $n \in \mathbb{N}$ we have that $\|x_n\| \leq 1$ and $|\Phi(x_n)| \to \infty$. Now, consider $a \in \ker\Phi$ and choose $y_n = a - \frac{x_n}{\Phi(x_n)}\Phi(a)$. Note, we can define y_n to ensure $\Phi(x_n) \neq 0$ since $|\Phi(x_n)| \to \infty$, there is only finitely many terms for which $\Phi(x_n) = 0$, so we can define a sequence which 'discards' these elements and still has the properties that $\|x_n'\| \leq 1$ and $|\Phi(x_n')| \to \infty$. Moreover, for each $n \in \mathbb{N}$ we have that $\Phi(y_n) = 0$, so $y_n \in \ker\Phi$. We also see that $y_n \to a$ by $|\Phi(x_n)| \to \infty$, but $a \notin \ker\Phi$. This shows $\Phi^{-1}(\{0\})$ is not closed. $\square$

We are now ready to complete the recap of material from last time by showing the norm in iii is complete.

1.7 Proposition. *Let A be a C^*-algebra without unit and $\tilde{A}$ be equipped with the norm induced by L, then $\tilde{A}$ is a Banach space.*

Proof. Let $(a, \lambda) \in \tilde{A}$ and $\pi_2(a, \lambda) = (0, \lambda)$, then we have that

$$\pi_2^{-1}(\{(0,0)\}) = (A, 0) \cong A.$$

It follows by the completeness of A that $\pi_2^{-1}(\{(0,0)\})$ is closed which implies that π_2 is a bounded linear map (it is essentially a bounded linear functional). Consequently, $\pi_1(a, \lambda) = (a, 0)$ is bounded since

$$\pi_1(a, \lambda) = (a, \lambda) - \pi_2(a, \lambda).$$

Suppose that $(a_n, \lambda_n) \in \tilde{A}$ is a Cauchy sequence. It follows from π_1 and π_2 being bounded linear maps that $\pi_1(a_n, \lambda_n)$ and $\pi_2(a_n, \lambda_n)$ are Cauchy. The completeness of $A \times \{0\}$ and $\{0\} \times \mathbb{C}$ implies that $a_n \to a$ and $\lambda_n \to \lambda$. Observe that,

$$\begin{aligned} \|L_{(b,\mu)}\| &= \|L_{(b,0)} + L_{(0,\mu)}\| \\ &\leq \|L_{(b,0)}\| + \|L_{(0,\mu)}\| \end{aligned}$$

Setting $b = a_n - a$ and $\mu = \lambda_n - \lambda$, we see that $L_{(a_n,\lambda_n)} \to L_{(a,\lambda)}$ in operator norm because by A and $\mathbb{C}$ being C^*-algebras, $\|L_a\| = \|a\|$ and $\|L_{(0,\mu)}\| = \|\mu\|$. $\square$

2 Properties of the spectrum

In the case of finite dimensional complex Hilbert spaces, we saw the spectrum is non-empty because the characteristic polynomial of a matrix has at least one root due to the Fundamental Theorem of algebra. Next, using a little complex analysis, we show that for $a \in A$, where A is a Banach algebra then $\sigma(a)$ is nonempty.

2.1 Theorem. *Let A be a Banach algebra with 1 [with the convention that $\|1\| = 1$]. We have*

(i) For $\|x\| < 1$, then $1 - x$ is invertible and $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$. Moreover,

$$\|(1-x)^{-1}\| \leq \frac{1}{1 - \|x\|}$$

and

$$\|(1-x)^{-1} - 1\| \leq \frac{\|x\|}{1 - \|x\|}$$

.

(ii) $G(A)$ is an open subset of A. More precisely, for each $a \in G(A)$, if $r = \frac{1}{\|a^{-1}\|}$, then $B_r(a) \subset G(A)$. Also, $G(A)$ is a group whose operations (multiplication and inverse) are continuous.

(iii) For each $a \in A$, $\sigma(a)$ is a compact subset of $\mathbb{C}$, and $r(a) \leq \|a\|$.

(iv) The function

$$\begin{aligned} R : \rho(a) &\to A \\ \lambda &\mapsto (a - \lambda 1)^{-1} \end{aligned}$$

is continuous and even analytic, i.e. for each $a \in A$, $\lambda_0 \in \rho(a)$, there is $r > 0$ such that for $\lambda \in B_r(\lambda_0)$,

$$R(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda_0)(\lambda - \lambda_0)^n$$

with some sequence $(a_n(\lambda_0))$ in A such that this series converges in norm of A.

Proof. (i) The convergence of the (Neumann) series follows from

$$\begin{aligned} \sum_{n=0}^{\infty} \|x^n\| &\leq \sum_{n=0}^{\infty} \|x\|^n \\ &= \frac{1}{1 - \|x\|} < \infty \end{aligned}$$

Let $y = \sum_{n=0}^{\infty} x^n$, then

$$\begin{aligned} y(1-x) &= (1-x)y \\ &= 1 + x + x^2 + x^3 + \cdots - (x + x^2 + x^3 + \cdots) = 1 \end{aligned}$$

So,

$$y = (1-x)^{-1}.$$

The norm of y is, by the above, bounded by

$$\|y\| \leq \frac{1}{1 - \|x\|}.$$

The second norm estimate follows from

$$\begin{aligned}\|y - 1\| = \|\sum_{n=1}^{\infty} x^n\| &\leq \|x\| \sum_{n=0}^{\infty} \|x\|^n \\ &= \frac{\|x\|}{1 - \|x\|}.\end{aligned}$$

(ii) Since A is submultiplicative, we get that $(a, b) \mapsto ab$ is continuous on $A \times A$. From part (i), we also see that $z \mapsto z^{-1}$ is continuous at 1, because if $x_n \to 0$ then $(1 - x_n)^{-1} \to 1$. The rest of the proof will be continued next time.

□