Lecture Notes from October 27, 2022

taken by Cristian Meraz

Last time

- Banach space clean up.
- Properties of the spectrum.

1.0.1 Warm-up (review)

We start by reviewing the Hahn-Banach theorem, a powerful theorem which allows us to easily compute with vectors via their "coordinates" (see first-day handout).

1.6 Theorem (Hahn-Banach, complex version)**.** *Given a complex normed space X, and a subspace Y with a bounded linear functional $f : Y \to \mathbb{C}$ such that $\|f\| = M$, there exists a linear functional $g : X \to \mathbb{C}$ such that $g|_Y = f$ and $\|g\| = M$.*

1.7 Remark. Note that although Banach is in the name of the theorem, it has nothing to do with Banach spaces or Banach algebras. In fact X need not be a normed space; it may have only a seminorm, or, simply a function $p : X \to \mathbb{R}$ satisfying

- $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$, and
- $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}^+$.

An important consequence is that the set of bounded linear functionals on a Banach algebra distinguishes between any two elements in the space.

1.8 Corollary. *If A is a Banach algebra and $a, b \in A$, $a \neq b$, then there exists a bounded linear functional $g : A \to \mathbb{C}$ such that $g(a) \neq g(b)$.*

Proof. Consider the subspace generated by $a - b$, $Y = \mathbb{C}(a - b)$. Let $f : Y \to \mathbb{C}$ be linear with

$$f(a - b) = \|a - b\|.$$

So we have

$$\|f\| = \sup_{\|a-b\| \leq 1} |f(a - b)| = \sup_{\|a-b\| \leq 1} \|a - b\| = 1.$$

By Hahn-Banach theorem, we know that there exists $g : X \to \mathbb{C}$ such that

$$g(a - b) = \|a - b\| \neq 0 \quad \Rightarrow \quad g(a) \neq g(b).$$

□

The next important result is a consequence of the Baire category theorem. Note that in the following, we can use the terms "nonmeager" or of "second-category" to describe a subset $B \subset X$ that is not a countable union of nowhere dense sets of X.

1.9 Theorem (Uniform boundedness or Banach-Steinhaus Theorem)**.** *Let* X *be a Banach space,* B *a set that is not a countable union of nowhere dense sets in* X*, and* Γ *a collection of bounded linear functionals. Suppose for any* $x \in B$, $\{\Lambda x : \Lambda \in \Gamma\}$ *is bounded. Then* T *is uniformly bounded, namely,*

$$\sup_{\Lambda \in \Gamma} \|\Lambda\| < \infty.$$

We prove the following special case.

1.10 Corollary. *If* A *is a Banach algebra, and* $(a_n)_{n\in\mathbb{N}}$ *a sequence such that for any* $f \in A'$, $(f(a_n))_{n\in\mathbb{N}}$ *is bounded, then*

$$\sup_{n\in\mathbb{N}} \|a_n\| < \infty.$$

- This is another example of using a property that holds for the coordinates to show it holds for the entire space.
- Here we are saying that "weak boundedness" of the sequence implies boundedness of the sequence; this does not hold when we are talking about convergence because weak convergence does not imply strong.

Proof. By Hahn-Banach, we know that for each $a \in A$, there exists an $f \in A'$ such that $f(a) = \|a\|$ and $\|f\| = 1$. We can see how a acts on the f's by writing

$$\|a\| = \sup_{\|f\|\leq 1} |f(a)|.$$

In particular if we define

$$i : A \to (A')', \qquad \underbrace{a \mapsto (f \mapsto f(a))}_{\text{canonical embedding}},$$

then we have

$$\|i(a)\| = \|a\|,$$

so i is an isometry. To see that the equality holds, note that the canonical embedding i maps a to a bounded linear functional $i(a)(\cdot)$ on A', which then takes a bounded linear functional f on A to its point evaluation $i(a)(f) = f(a) \in \mathbb{C}$. Now when we write the norm

$$\|i(a)\| = \sup_{\|f\|\leq 1} |i(a)(f)| = \sup_{\|f\|\leq 1} |f(a)|,$$

this is the equality we have coming from Hahn-Banach.

Moreover, if we assume that $(f(a_n))_{n\in\mathbb{N}} = (i(a_n)(f))_{n\in\mathbb{N}}$ stays bounded for each $f \in A'$, then since A' is a Banach space, by uniform boundedness we have

$$\sup_{n\in\mathbb{N}} \|i(a_n)\| = \sup_{n\in\mathbb{N}} \|a_n\| < \infty.$$

$\square$

1.0.2 Back to proof from last time

We were proving the following theorem:

1.11 Theorem. *Let* A *be a Banach algebra with unit 1 and* $\|1\| = 1$*. We have:*

(i) For $\|x\| < 1$*, the element* $1 - x$ *is invertible, and*

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

Moreover,

$$\|(1-x)^{-1}\| \leq \frac{1}{1-\|x\|}$$

and

$$\|(1-x)^{-1} - 1\| \leq \frac{\|x\|}{1-\|x\|}.$$

(ii) $G(A)$ *is an open subset of* A*. More precisely, for each* $a \in G(A)$*, if* $r = \|a^{-1}\|^{-1}$*, then*

$$B_r(a) \subset G(A).$$

Also, $G(A)$ *is a group whose operations (multiplication and inversion) are continuous.*

(iii) For each $a \in A$*,* $\sigma(a)$ *is a compact subset of* $\mathbb{C}$ *and* $r(a) \leq \|a\|$*.*

(iv) The function

$$R : \rho(A) \to A, \quad \lambda \mapsto (a - \lambda 1)^{-1},$$

is continuous and even analytic, that is, for each $a \in A$*,* $\lambda_0 \in \rho(a)$ *, there is* $r > 0$ *such that for* $\lambda \in B_r(\lambda_0)$*,*

$$R(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda_0)(\lambda - \lambda_0)^n,$$

with some sequence $(a_n(\lambda_0))$ *in* A *such that the series converges in the norm of* A*.*

Proof of (ii). Let $a \in G(A)$, and assume $\|x\| < \|a^{-1}\|^{-1}$. Then $a - x = a(1 - a^{-1}x)$ and thus

$$\|a^{-1}x\| \leq \|a^{-1}\|\|x\| < 1.$$

From (i), the element $1 - a^{-1}x \in G(A)$, and since $a \in G(A)$, we have

$$a(1 - a^{-1}x) = a - x \in G(A).$$

This works for any such x, so the claim follows. To see this, let $y \in B_r(a) \equiv \{z \in A : \|z - a\| < r\}$ with $r = \|a^{-1}\|^{-1}$. Then defining $x = a - y$ gives

$$\|x\| < \|a^{-1}\|^{-1},$$

and thus

$$y = a - x \in G(A).$$

To understand that $G(A)$ is a topological group, we need to show continuity of the group operations: the product and the inversion. We already showed continuity of the product in A, hence this also holds for the subset $G(A)$. We need to show continuity of the inversion.

To this end, let $a \in G(A)$ and $\|x\| < \|a^{-1}\|^{-1}$. Then $a - x \in G(A)$ and

$$(a - x)^{-1} = (1 - a^{-1}x)a^{-1}.$$

We know that the right-hand side is (sequentially) continuous at $x = 0$, and so is the left-hand side. Thus $a \mapsto a^{-1}$ is a continuous map at each $a \in G(A)$.

□

Proof of (iii). We know that $R : \rho(a) \to A,\ \lambda \mapsto (a - \lambda 1)^{-1}$ is continuous at $\lambda \in \rho(a)$ by (ii), so $\rho(a)$ is open and hence $\sigma(a)$ is closed. For $|\lambda| > \|a\|$, we get

$$\lambda 1 - a = \lambda(1 - \lambda^{-1}a)$$

and thus

$$\|\lambda^{-1}a\| = \frac{\|a\|}{|\lambda|} < 1.$$

From (i), we have

$$1 - \lambda^{-1}a \in G(A),$$

so also

$$\lambda(1 - \lambda^{-1}a) = \lambda 1 - a \in G(A),$$

meaning $\lambda \in \rho(a)$. We have shown that $\lambda \in \mathbb{C}$ with the property $|\lambda| > \|a\|$ implies $\lambda \in \rho(a)$; in symbols,

$$\{\lambda \in \mathbb{C} : \||\lambda| > \|a\|\} \subset \rho(a).$$

Then taking complements we have

$$\sigma(a) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\},$$

and this implies

$$r(a) \equiv \sup_{\lambda \in \sigma(a)} |\lambda| \leq \|a\|.$$

Hence we see that $\sigma(a)$ is bounded in $\mathbb{C}$. By closedness, $\sigma(a)$ is compact. □

Proof of (iv). Let $\lambda_0 \in \rho(a)$, $r = \|R(\lambda_0)\|^{-1}$. For $|\lambda - \lambda_0| < r$, we get $\lambda \in \rho(A)$ from (ii). The proof of (ii) gives an expression for $R(\lambda)$

$$\begin{aligned} R(\lambda) &= (\lambda 1 - a)^{-1} \\ &= (\underbrace{(\lambda_0 1 - a)}_{A} - \underbrace{(\lambda_0 - \lambda)1}_{X})^{-1} \\ &= \underbrace{(1 - R(\lambda_0)(\lambda_0 - \lambda))^{-1}}_{(1 - A^{-1}X)^{-1}} \underbrace{R(\lambda_0)}_{A^{-1}}. \end{aligned}$$

(Note we use the uppercase A to highlight the similarities between what we have here and when we considered $(a-x)^{-1}$. Not to be confused with the name of the Banach algebra.) We obtain the Neumann series

$$\begin{aligned} R(\lambda) &= \left(\sum_{n=0}^{\infty} R(\lambda_0)^n(\lambda_0-\lambda)\right) R(\lambda_0) \\ &= \sum_{n=0}^{\infty} R(\lambda_0)^{n+1}(-1)^n(\lambda-\lambda_0)^n \end{aligned}$$

and the series converges using

$$\|R(\lambda_0)(\lambda-\lambda_0)\| = |\lambda-\lambda_0|\|R(\lambda_0)\| < 1.$$

So, we have shown (iv). □

It remains to show that the spectrum is nonempty to finish elementary properties.