# Lecture Notes from October 27, 2022

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**Last time**

- Banach space clean-up
- Properties of the spectrum

**Warm up:**

**1.2 Theorem.** Hahn-Banach, Complex Version

*Given a complex normed space* $X$*, and a subspace* $Y$ *with a bounded linear functional* $f : Y \mapsto \mathbb{C}$ *such that* $\|f\| = M$ *then there is a linear functional* $g : X \mapsto \mathbb{C}$ *such that* $g|_y = f$ *and* $\|g\| = M$

**1.3 Corollary.** *If* $A$ *is a Banach algebra and* $a, b \in A, a \neq b$ *then there is a bounded linear functional* $g : A \mapsto \mathbb{C}$ *such that* $g(a) \neq g(b)$

*Proof.* Consider $Y = \mathbb{C}(a - b)$, let $f : Y \mapsto \mathbb{C}$ be linear with $f(a, b) = \|a - b\|$ so $\|f\| = 1$, by Hahn-Banach we get $g : A \mapsto \mathbb{C}, g$ bounded such that the $g(a - b) = \|a - b\| \neq 0$ so $g(a) - g(b) = \|a - b\|$

□

**1.4 Theorem.** Uniform Boundedness

*Let* $X$ *be a Banach Space,* $B$ *be a set that is not a countable union of nowhere dense sets in* $X$*, and* $\Gamma$ *a collection of bounded linear functionals, and for any* $x \in B, \{\Lambda x : \Lambda \in \Gamma\}$ *is bounded, then* $\Gamma$ *is uniformly bounded, ie*

$$\sup_{\Lambda \in \Gamma} \|\Lambda\| < \infty$$

**1.5 Corollary.** *If* $A$ *is a Banach algebra, and* $(a_n)$ *a sequence such that for any* $f \in A', (f(a_n))_{n \in \mathbb{N}}$ *is bounded, then* $\sup_{n \in \mathbb{N} < \infty}$

*Proof.* By Hahn-Banach, we know that for each $a \in A$ there is $f \in A', \|f\| = 1, f(a) = \|a\|$ Hence, $\|a\| = \sup_{\|f\| \leq 1} |f(a)$ We define

$$i : A \mapsto (A')'$$

$$a \mapsto (f \mapsto f(a))$$

then $\|i(a)\| = \|a\|$, so $i$ is an isometry.

Moreover, $i(a_n)(f) = f(a_n)$ and if we assume that $(f(a_n))$ stays bounded for each $f \in A', A'$ Banach, then by uniform boundedness, $\sup_n \|i(a_n)\| = \sup_n \|a_n\| < \infty$

□

Now let's recall the properties of the spectrum from last time and finish proving them.

*Proof.* i This property was proved in the last class

ii Let $a \in G(A), \|x\| < \|a^{-1}\|^{-1}$, then $a - x = a(1 - a^{-1}x)$ and $\|a^{-1}x\| \leq \|a^{-1}\|\|x\| < 1$ from (i) we have have $1 - a^{-1}x \in G(A)$ and from $a \in G(A), a(1 - a^{-1}x) = a - x \in G(A)$. This works for any such $x$, so the claim follows because if $y \in B_r(a)$, with $r = \|a^{-1}\|^{-1}$ then defining $x = a - y$ gives $\|x\| < \|a^{-1}\|^{-1}$ and $y = a - x \in G(A)$

To understand that $G(A)$ is a topological group, we need to show continuity of the group operations. We already showed continuity of the product in $A$, hence this also holds for the subset $G(A)$. Now, we need to continuity of the inversion.

Let $a \in G(A), \|x\| < \|a^{-1}\|^{-1}$, then $a - x \in G(A)$ and $(a - x)^{-1} = (1 - a^{-1}x)^{-1}a^{-1}$

We know the right hand side is continuous at $x = 0$ and so is the left hand side and thus $a \mapsto a^{-1}$ is continuous at each $a \in G(A)$

iii We have that

$$R : \rho(a) \mapsto A$$
$$\lambda \mapsto (a - \lambda 1)^{-1}$$

is continuous at $\lambda \in \rho(a)$ by (ii), so $\sigma(a)$ is closed.

For $\lambda > \|a\|$, we get $\lambda 1 - a = \lambda(1 - \lambda^{-1}a)$ and thus $\|\lambda^{-1}a\| = \frac{\|a\|}{|\lambda|} < 1$

From (i), we have $1 - \lambda^{-1}a \in G(A)$, so also $\lambda 1 - a \in G(A)$, meaning $\lambda \in \rho(a)$.

This implies that $r(a) \leq \|a\|$ and we $\sigma(a)$ is bounded in $\mathbb{C}$. By closeness, $\sigma(a)$ is compact.

iv Let $\lambda_0 \in \rho(a), r = \|R(\lambda_0)\|^{-1}$, for $|\lambda - \lambda_0| < r$ we get $\lambda \in \rho(a)$ from (ii) The proof of (ii) gives us an expression for $R(\lambda)$,

$$R(\lambda) = (\lambda 1 - a)^{-1} = (\underbrace{(\lambda_0 1 - a)}_{A} - \underbrace{(\lambda_0 - \lambda)}_{X})^{-1}$$

$$= \overbrace{(1 - R(\lambda_0)(\lambda_0 - \lambda))^{-1}}^{(1 - A^{-1})}\overbrace{R(\lambda_0)}^{A^{-1}}$$

$$= (\sum_{n=0}^{\infty} R(\lambda_0)^n(\lambda_0 - \lambda)^n)R(\lambda_0) = \sum_{n=0}^{\infty} R(\lambda_0)^{n=1}(-1)^n(\lambda_0 - \lambda)^n$$

and the series converges using $\|R(\lambda_0)(\lambda - \lambda_0\| = |\lambda - \lambda_0|\|R(\lambda_0)\| < 1$ and so we have shown (iv)

□