Lecture Notes from November 1, 2022

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Last Time

- Hahn Banach over $\mathbb{C}$
- Uniform Boundedness
- Properties of the spectrum

Warm up:

Given $f : \mathbb{D} \to \mathbb{C}$, f analytic on $\mathbb{D}$, $\overline{B}_1(0) \subset \mathbb{D}$, $\mathbb{D}$ open, then f has a uniformly convergent power series on $\overline{B}_r(0)$ for any $r < 1$ given by

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-itn}\, dt\; z^n.$$

First, we want to show the series converges. On $\overline{B}_r(0)$, f is by assumption continuous, hence bounded since $\overline{B}_1(0)$ is compact.

Consequently,

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-itn}\, dt$$

satisfies

$$\begin{aligned} |c_n| &\le \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it}) e^{-itn}|\, dt \\ &\le \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})| \cancelto{1}{|e^{-itn}|}\, dt && \text{(by Hölder)} \\ &\le \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|\, dt \le \|f\|_\infty. \end{aligned}$$

Since $z \in \overline{B}_r(0)$, $|z| \le r < 1$. Hence, by Hölder again,

$$\sum_{n=0}^{\infty} |c_n z^n| \le \sum_{n=0}^{\infty} |c_n||z|^n \le \sum_{n=0}^{\infty} \|f\|_\infty r^n < \infty.$$

By the Weierstraβ M-test, $\sum_{n=0}^{\infty} c_n z^n$ is uniformly convergent on $\overline{B}_r(0)$ for any $r < 1$.

Next, since Fubini-Tonelli allows us to interchange integrals and sums, we observe

$$\begin{aligned}
g(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-itn} \, dt \; z^n \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \sum_{n=0}^{\infty} e^{-itn} \, dt \; z^n \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \sum_{n=0}^{\infty} e^{-itn} z^n \, dt && \text{(Geometric Series)} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \frac{1}{1 - e^{-it} z} \, dt.
\end{aligned}$$

We have that g is analytic (as g is a uniformly convergent limit of polynomials) on each $\overline{B}_r(0)$, $r < 1$. Also note that for $r = 1$, we get the Fourier series of $f(e^{it})$, which is convergent in L^2. Using Dominated Convergence of Fourier coefficients in l^2, i.e.

$$\hat{g}_r(n) = r^n c_n$$

then $r \to 1$ gives $\hat{g}_r \to (c_n)$ in l^2. Consequently, as $r \to 1$,

$$g_r(e^{it}) = \sum_{n=0}^{\infty} \hat{g}_r(n) \, e^{itn} \xrightarrow{L_2} f(e^{it})$$

We conclude, g is a power series converging to f in L^2.

Preparing for Banach-Mazur

To deduce properties of the spectrum, we use complex analysis.

0.0 Theorem. *Let* $0 < r < R$, $\Omega = \{z \in \mathbb{C} : r < |z| < R\}$ *and* $f : \Omega \to \mathbb{C}$ *analytic, then* f *has a Laurent series*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

with uniform convergence on compact subsets of Ω.

And for any $r < \rho < R$,

$$a_n = \frac{1}{2\pi\rho} \int_0^{2\pi} f(\rho e^{it}) e^{-int}\, dt.$$

Proof. For simplicity, we assume f has a Laurent series expression[1], then we show a_n is given by this integral. By uniform convergence of series, we may integrate term-by-term,

$$\begin{aligned}
\int_0^{2\pi} f(\rho e^{it}) e^{-int}\, dt &= \int_0^{2\pi} \left(\sum_{m=-\infty}^{\infty} a_m (\rho e^{it})^m \right) e^{-int}\, dt \\
&= \sum_{m=-\infty}^{\infty} \int_0^{2\pi} a_m \rho^m e^{i(m-n)t}\, dt \\
&= \sum_{m=-\infty}^{\infty} a_m \rho^m \int_0^{2\pi} e^{i(m-n)t}\, dt \\
&= a_n \rho^n \int_0^{2\pi} e^{i(0)t}\, dt \qquad \left(m \neq n \implies \textstyle\int_0^{2\pi} e^{i(m-n)t} = 0\right) \\
&= 2\pi a_n \rho^n
\end{aligned}$$

Diving by $2\pi\rho^n$ gives the claimed expression. □

As a consequence, we get Liouville's theorem.

0.1 Theorem. *Let* $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *be bounded and analytic. Then* f *is constant.*

Proof. For $\rho \in (0, \infty)$, we get

$$f(z) = \sum_{m=-\infty}^{\infty} a_n z^n$$

with $a_n = \frac{1}{2\pi\rho^n} \int_0^{2\pi} f(\rho e^{it}) e^{-int}\, dt$. Then since $\|f\|_\infty = \sup_{z \in \mathbb{C} \setminus \{0\}} |f(z)| < \infty$,

$$\begin{aligned}
|a_n| &\leq \frac{1}{2\pi\rho^n} \int_0^{2\pi} \|f\|_\infty \cancelto{1}{|e^{-int}|}\, dt \\
&\leq \frac{\|f\|_\infty}{\rho^n}.
\end{aligned}$$

If $n < 0$, letting $\rho \to 0$ shows $a_n = 0$. If $n > 0$, $\rho \to \infty$ gives $a_n = 0$. Therefore $f(z) = a_0$, so f is constant. □

[1] To see why analytic functions on an annulus have a Laurent series expression, see Chapter 5 Section 1.1.3: The Laurent Series in *Complex Analysis* by Lars Ahlfors (Third Edition).

We need one more lemma to prepare the main result on the spectrum.

0.2 Lemma. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}^+$ *with* $0 \le a_{n+m} \le a_n a_m$. *Then* $({a_n}^{\frac{1}{n}})_{n\in\mathbb{N}}$ *converges to* $\inf_{n\in\mathbb{N}} {a_n}^{\frac{1}{n}}$.

Proof. Let $a = \inf_{n\in\mathbb{N}} {a_n}^{\frac{1}{n}}$. Choose $\epsilon > 0$. Then we can find $N \in \mathbb{N}$ such that ${a_N}^{\frac{1}{N}} < a + \epsilon$.
Let $b = \max\{1, a_1, a_2, \ldots, a_{N-1}\}$ and write $n = kN + r$ with $r \in \{0, 1, 2, \ldots, N-1\}$.
Then

$$
\begin{aligned}
{a_n}^{\frac{1}{n}} = {a_{kN+r}}^{\frac{1}{n}} \le (a_N^k a_r)^{\frac{1}{n}} &\le (a+\epsilon)^{\frac{kN}{n}} b^{\frac{1}{n}} \\
&= (a+\epsilon)^{1-\frac{r}{n}} b^{\frac{1}{n}} \\
&= (a+\epsilon)(a+\epsilon)^{-\frac{r}{n}} b^{\frac{1}{n}}
\end{aligned}
$$

As $(a+\epsilon)^{-\frac{r}{n}} \to 1$ and $b^{\frac{1}{n}} \to 1$ as $n \to \infty$, we have that by convergence of factors, for all sufficiently large n,

$$
{a_n}^{\frac{1}{n}} \le a + 2\epsilon.
$$

Since ϵ was arbitrary, ${a_n}^{\frac{1}{n}} \to a$. □