Lecture Notes from November 1, 2022

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Last time

- Hahn Banach Theorem over $\mathbb{C}$
- Uniform Boundedness
- Properties of the Spectrum of an operator

Warm up:

Given an analytic function $f : D \to \mathbb{C}$ for some domain D containing the closed unit disc $\overline{B_1(0)}$ we know from complex analysis that f has a uniformly convergent power series on $\overline{B_r(0)}$ for any $0 < r < 1$. Recall the Cauchy Integral formula, (aka Cauchy's Differentiation Formula according to Wikipedia) for an analytic function on an open domain containing the origin,

$$f^{(n)}(0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta^{n+1}}d\zeta,$$

where γ is any smooth closed curve in the domain that circles around the origin once. In our case we can calculate the integral with the parametrization $\zeta = e^{it}$, so $d\zeta = ie^{it}dt$ Thus,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}z^n = \sum_{n=0}^{\infty}(\frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta^{n+1}}d\zeta)z^n = \sum_{n=0}^{\infty}(\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-itn}dt)z^n.$$

I think we are trying to prove that the series converges on the closed unit disc.

$\overline{B_1(0)}$ is compact and f is continuous, so f is bounded on the closed unit disc. Consequently, by Hlder, $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-itn}dt$ satisfy

$$|c_n| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|dt \leq \|f(e^{it})\|_\infty = M.$$

For $|z| \leq r$, we have $|c_n z^n| \leq Mr^n$ and since $r < 1$, $\sum Mr^n$ converges. Hence, $\sum c_n z^n$ converges uniformly converges on $\overline{B_r(0)}$ by the Weier-Strauss M test.

This implies we can interchange integral and summation so, for $|z| \leq r < 1$,

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-itn} dt z^n \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \sum_{n=0}^{\infty} e^{-itn} z^n dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{-it})}{1 - e^{-it} z} dt,
\end{aligned}
$$

where the sum converges since it is a geometric series for a fixed z.

Since we already assumed f is analytic, we have $\lim_{r\to 1} f(re^{it}) = f(e^{it})$. At this moment our discussion seems tautological. Like we have only been going in circles (pun intended), but we have already determined something. The assumption that f be analytic is too strong. We were able to recover everything about f on $\overline{B_1(0)}$ by only knowing the values of f on $\mathbb{T}$. Even so, let us see what we can determine if we only are given a continuous function, g, defined on $\mathbb{T}$. The Fourier coefficients of g are defined as above. That is, $c_n = \frac{1}{2\pi} \int_0^{2\pi} g(e^{it}) e^{-itn} dt$. We can define a power series on $\mathbb{T}$ by $\sum c_n z^n$, where $z \in \mathbb{T}$ iff $|z| = 1$.

As we have already shown, continuity of g implies $|c_n| \leq ||g(e^{it})||_\infty$. Thus, for $r < 1$ as above, $\sum c_n z^n$ converges uniformly for $z \in \overline{B_r(0)}$. Hence, we can differentiate term by term, so we have "extended" g to an analytic function $\tilde{g} = \sum c_n z^n$ on $\overline{B_r(0)}$. We still need to show $\tilde{g}$ converges to g in some sense. In class we showed $\lim_{r\to 1} \tilde{g}(re^{it}) = g(e^{it})$ for almost every $t \in [0, 2\pi]$ as follows. For $r = 1$, g has a Fourier series that converges in $L^2(\mathbb{T})$. Hence, $(c_n) \in l^2$. Instead of thinking of g being defined on the circles with radius less than one, consider a family of functions $g_r : \mathbb{T} \to \mathbb{C}$ defined by $g_r(e^{it}) = \sum r^n c_n e^{it}$. Now, using dominated convergence of the Fourier coefficients in l^2 we have $\lim_{r\to 1} \hat{g_r}(n) = \lim_{r\to 1} r^n c_n = c_n$. That is, $(g_r)_n \to (c_n)$ in l^2. Thus, $g_r \to g(e^{it})$ in L^2. Therefore, $\lim_{r\to 1} g(re^{it}) = g(e^{it})$ for almost every $t \in [0, 2\pi]$.

In fact, a stronger result is true, $\lim_{r\to 1} \tilde{g}(re^{it}) = g(e^{it})$ uniformly for all $t \in [0, 2\pi]$. This follows from Poisson's Theorem, (Davidson and Donsig in their book Real Analysis and Applications: Theory in Practice pg 341). In their proof, they use properties of the Poisson kernel

$$P(r, t) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(t) + r^2}.$$

Their result is in terms of harmonic analysis where the function on $\mathbb{T}$ is real, however, the result extends to complex functions by defining $g(e^{it}) = u(t) + iv(t)$ and applying Poisson's Theorem to u and v. It still remains to show that u and v are harmonic conjugates.

There are some more complex analysis theorems we will need to prove properties of the spectrum.

1.6 Theorem. *Let* $0 < r < R$, $\Omega = \{z \in \mathbb{C} : r < |z| < R\}$ *and* $f : \Omega \to \mathbb{C}$ *be analytic. Then,* f *has a Laurent series* $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ *that converges uniformly on compact subsets of* Ω *and for any* $r < \rho < R$,

$$a_n = \frac{1}{\rho^n} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{it}) e^{-itn} dt.$$

Proof. That an analytic function has a uniformly convergent Laurent series on compact subsets of Ω can be seen immediately from the fact that $f(z)$ has a Taylor series for $|z| < R$ and $f(1/z)$ has a Taylor series for $|z| > r$. As with the warm-up let us use uniform convergence to integrate term by term.

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it})e^{-itn}dt &= \sum_{m\in\mathbb{Z}} a_m\rho^m \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}dt \\ &= \sum_{m\in\mathbb{Z}} a_m\rho^m\delta_{mn} = a_n\rho^n,\end{aligned}$$

giving the formula. □

1.7 Corollary. *(Louiville's Theorem) Let* $f : \mathbb{C} - \{0\} \to \mathbb{C}$ *be bounded and analytic. Then* f *is constant.*

Proof. For $\rho \in (0, \infty)$ we have $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ with

$$a_n = \frac{1}{\rho^n}\frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it})e^{-itn}dt.$$

Since f is bounded, $||f||_\infty = \sup_{z\neq 0}|f(z)| \leq M < \infty$. Thus, by Hlder,

$$|a_n| \leq \frac{1}{\rho^n}\frac{1}{2\pi}\int_0^{2\pi} M dt \leq \frac{M}{\rho^n}.$$

For $n > 0$ let ρ approach 0 and for $n < 0$ let ρ approach ∞. Therefore, $a_n = 0$ for all $n \neq 0$, so $f(z) = a_0$ for all $z \neq 0$. □

We are still setting up the pieces to be used to prove the spectrum is nonempty.

1.8 Lemma. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}^+$ *that satisfies* $0 \leq a_{n+m} \leq a_n a_m$. *Then,* $(a_n)_n^{1/n}$ *converges to* $\inf_{n\in\mathbb{N}} a_n^{1/n}$.

Proof. Let $a = \inf_{n\in\mathbb{N}} a_n^{1/n}$ and let $\epsilon < 0$. Hence, there exists $N \in \mathbb{N}$ such that $a_N^{1/N} < a + \epsilon$. Let $b = \max 1, a_1, ..., a_{N-1}$ and let $n = Nk + r$. Then,

$$\begin{aligned}a_n^{1/n} = a_{Nk+r}^{1/n} &\leq (a_N^k a_r)^{1/n} \\ &\leq (a+\epsilon)^{kN/n} b^{1/n} \\ &= (a+\epsilon)^{1-r/n} b^{1/n} \\ &= (a+\epsilon)((a+\epsilon)^{-r})^{1/n} b^{1/n} \to a+\epsilon \text{ as } n \text{ approaches } \infty.\end{aligned}$$

The limit evaluation on the last step follows from the fact $c^{1/n}$ converges to 1 for all $c > 0$ as n approaches ∞. Since ϵ was arbitrary, we conclude $a_n^{1/n}$ converges to a. □