Lecture Notes from November 3, 2022

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Last Time

- Fourier Series VS Power Series for analytic functions
- Liouville's Theorem
- Asymptotic rate of growth for sequences

Warm up:

For $0 \leq a_{n+m} \leq a_n a_m$, then $a_n^{\frac{1}{n}} \to \inf_n a_n^{\frac{1}{n}}$.
Suppose that the equality holds, i.e, $a_{n+m} = a_n a_m$, then what do I know about a's?

Then a: $\mathbb{N} \to \mathbb{R}^+$ is semi-group homomorphism.
All of this is just determined by a_1 and $a_n = a_1^n$.

This is precisely an exponential growth.

So, for $a_{n+m} < a_n a_m$, one can argue its sub exponential growth.

i.e, the rate at which this thing grows is inf $a^{1/n} = 0$ or $\lim_{n\to\infty} a^{1/n} = 0$

1.2 Theorem. *Let a ∈ A, A a Banach algebra, then*

i) $\sigma(a) \neq \emptyset$

ii) r(a) $= \lim_{n\to\infty} ||a^n||^{1/n} = inf_n \, ||a^n||^{1/n}$

Proof. :

i) Assume $\sigma(a) \neq \phi$

Then resolvent set $\rho(a) = \mathbb{C}$, R : $\mathbb{C} \to$ A is analytic in $\mathbb{C}$.

For $|\lambda| > ||a||$, we get,

$$
\begin{aligned}
||R(\lambda)|| &= ||(\lambda 1 - a)^{-1}|| \\
&= |\lambda|||(1-a)^{-1}|| \\
&\leq \frac{1}{1 - \frac{1}{|\lambda|}||a||} \\
&= \frac{1}{|\lambda| - ||a||}
\end{aligned}
$$

Hence, $||R(\lambda)||$ is uniformly bounded in a set and R is bounded on $\{\lambda \in \mathbb{C} : |\lambda| > ||a||\}$

On the other hand, R is continuous on any closed disk in $\mathbb{C}$, so R is bounded on all of $\mathbb{C}$

Let $f \in A'$, then $f \circ R : \mathbb{C} \to \mathbb{C}$ is bounded and analytic, consequently it is constant.

$$
\begin{aligned}
\text{So,} \quad f \circ R(\lambda) &= f \circ R(0) \\
&= f(-a^{-1})
\end{aligned}
$$

Here, $\lambda = 0$ and the resolvent at 0 is $-a^{-1}$.

Since f : $f \in$ A' , the set of all f distinguish between any two elements in A.

So, we know, R $(\lambda) = -a^{-1}$ for any λ

But then,

$$
\begin{aligned}
\text{for } \lambda \in \mathbb{C}, \ \lambda 1 - a &= \lambda 1 - (a^{-1})^{-1} \\
&= (R(\lambda))^{-1} \\
&= (a^{-1})^{-1} \\
&= -a
\end{aligned}
$$

Contradiction.
This should hold for every λ and not just for $\lambda = 0$. This shows that $\sigma(a) \neq \emptyset$

□

Proof.
ii) Let $s(a) = \inf ||a^n||^{1/n}$,
then by our lemma (and sub-multiplicativity of norm), $s(a) = \lim_{n\to\infty} ||a^n||^{1/n}$.

Here, our claim is that if $|\lambda| > s(a)$, then it must be the element of s(a).

Now, we show if $|\lambda| >$ s(a), then $\mathcal{X} \in \rho(a)$.
where s(a) = candidate for the spectral radius and $\rho(a)$ = resolvent set.

To this end, note that

$$\limsup_{n} ||(\lambda^{-1}a)^n||^{1/n} = \lim_{n\to\infty} \frac{1}{|\lambda|}||a^n||^{1/n}$$
$$< 1$$

By asymptotic bound, $\sum_{n=0}^{\infty} \lambda^{-n} a^n (< 1 - \varepsilon^n)$ converges.

It is Geometric series and we recall that if the series converges, then

$$R(\lambda) = (\lambda 1 - a)^{-1}$$
$$= \frac{1}{\lambda}(1 - \lambda^{-1}a)^{-1}$$

We then get,
$R\ (\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n$

and hence, $\lambda \in \rho(a)$

By comparing $\sigma(a)$ and $\rho(a)$, we see that

$$r(a) = sup\{|\lambda| : \lambda \in \sigma(a)\}$$
$$So, \quad r(a) \leq s(a) \qquad --- \qquad i)$$

Next, we want to show that equality holds between s(a) and r(a).

Let $\Omega = \{z \in \mathbb{C} > r(a)\} \subset \rho(a)$
(Here, r > r(a). Since $\sigma(a) \neq \emptyset$, r(a) is some number. So, we will look at all the elements in $\mathbb{C}$ and derive s(a) > r(a)).

For each continuous linear functional f ∈ A', we have f o R : $\Omega \to \mathbb{C}$ is analytic and it has a series expansion.

So, f o R (z) = $\sum_{n\in\mathbb{Z}} c_n z^n$

but, for $|z| > ||a||$, we know,
$R(z) = \sum_{n=0}^{\infty} z^{-n-1} a^n$

Comparing coefficients gives that

$$c_n = 0 \quad for \ \ n \geq 0$$
$$and \ \ c_{-1-n} = f(a^n) \ \ for \ \ n \geq 0$$

Here, $||a^n||$ is bounded and if we divide by z, it converges to 0.

By our choice of z, $\lim_{n\to\infty} f(a^n)z^{-n} = 0$
(Here, we fix our z. Then, a^n is a convergent sequence in n. Thus, it has a maximum value or sup somewhere. So, for every choice of $f(a^n)$, we will get a sequence of functionals i.e., sequence of operators which are bounded and for every z, we have a bounded sequence of operators)

Now using Banach - Stienhaus, we can see that for $z \in \Omega$, $(z^{-n}a^n)n \in \mathbb{N}$ is uniformly bounded (as a sequence).

Hence, the norms of all of these has some finite sup.

So, there is $C > 0$ with $||a^n|| \leq C|z|^n$

This is true for all $n \in \mathbb{N}$.

Now taking nth root and letting $n \to \infty$, we have $s(a) \leq \lim_{n\to\infty} C^{1/n}|z|$

This works for each z with $|z| > r(a)$

Taking infimum over all such z gives $s(a) \leq r(a)$ - - - ii)

Since from i) and ii) we get, $r(a) \leq s(a)$ and $s(a) \leq r(a)$, we have,

Thus, we have, $r(a) = s(a) = \inf||a^n||^{1/n}$

□

By doing this theorem we connected spectral radius (invertibility) and asymptotic rate of growth of norms of powers of a.

1.3 Remark. We observe that $r(a) = \lim_{n\to\infty} ||a^n||^{1/n}$ relates algebraic and topological quantities without assuming C^*-algebra structure.

1.4 Theorem. *(Gelfand-Mazur): Let A be a Banach algebra with unit 1, in which each element $a \neq 0$ is invertible then $A \cong \mathbb{C}$ i.e, dim A = 1*

Proof. : Let $a \in A$.

Then $\sigma(a) \neq \emptyset$ and $\lambda 1 - a = 0$ because a is multiple of identity.

By theorem on the spectrum, there is $\lambda \in \sigma(a)$, so $\lambda 1 - a \notin G(A)$.

But by our assumption, $\lambda 1 - a = 0$, then, a $= \lambda 1$

So if we define a map $\eta : \mathbb{C} \to A$ by $\eta(\lambda) = \lambda 1$.

Thus, we can see that A $\cong \mathbb{C}$

□

In C*-algebra, we find a more direct relation between the spectral radius and norm.

1.5 Lemma. *Let A be a* C^**-algebra then,*

i) $\sigma(a^*) = \sigma(a) = \{z : z \in \sigma(a)\}$

ii) If a is normal, r(a) $= \|a\|$

iii) For a $\in$ *A,* $\|a\| = \sqrt{r(a * a)}$

Proof.

i) We know that a - $\lambda 1$ is invertible in $\tilde{A}$ iff $(a - \lambda 1)^* = a^* - \overline{\lambda} 1)$

since $\overline{\lambda} \in \sigma(a^*)$ iff $\lambda \in \overline{\sigma(a)}$

This gives the claimed. □