Lecture Notes from Novermber 3, 2022

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0 Warm-Up

Recall that $0 \leq a_{n+m} \leq a_n a_m$ implies $a_n^{1/n} \to inf a_n^{1/n}$ for $a_n \in \mathbb{R}$. Looking at the edge case of $a_{n+m} = a_n a_m$, we see that $a : \mathbb{N} \to \mathbb{R}^+$ is a semigroup homomorphism determined by $a_n = a_1^n$.

1 The Spectral Theorem

1.0.1 Theorem. *Let* $a \in A$*,* A *a Banach Algebra. Then we have the following results for the spectrum:*

(i) $\sigma(a) \neq \emptyset$

(ii) $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_n \|a^n\|^{1/n}$

Proof. Assume $\sigma(a) =$ Then $\rho(a) = \mathbb{C} - \sigma(a) = \mathbb{C}$. $R : \mathbb{C} \to A$ s.t. $R(\lambda) = (\lambda 1 - a)^{-1}$ is analytic in $\mathbb{C}$, so for $\lambda > \|a\|$ we get $\|R(\lambda)\| = \|(\lambda 1 - a)^{-1}\| = |\lambda| \|(1 - \lambda^{-1} a)^{-1}\| = |\lambda|^{-1} \|(1 - \lambda^{-1} a)\| \leq \lambda^{-1} \frac{1}{1 - \frac{1}{|\lambda|}\|a\|}$

Hence R is bounded on $\{\lambda \in \mathbb{C} : |\lambda| \geq \|a\|\}$. On the other hand, R is continuous on any closed disk in $\mathbb{C}$, so R is bounded on all of $\mathbb{C}$. Now let $f \in A'$ (the dual of A). Then $f \circ R : \mathbb{C} \to \mathbb{C}$ is bounded analytic, hence it is constant so $f \circ R(\lambda) = f \circ R(0) = f(-a)$

Since $\{f : f \in A'\}$ distinguishes between any 2 elements in A, we have $R(\lambda) = -a^{-1}$. But then $\lambda 1 - a = R(\lambda)^{-1} = (-a)^{-1} = -a$ for each $\lambda \in \mathbb{C}$ for each $\lambda \in \mathbb{C}$. Contradiction! This proves (i)

Now we prove item (ii). Let $s(a) = \inf_n \|a^n\|^{1/n}$, then as in the warm-up, we have $s(a) = \lim_{n\to\infty} \|a_n\|^{1/n}$. We show if $|\lambda| s(a)$, the $\lambda \in \rho(a)$. Note that $\limsup_n \|(\lambda^{-1} a)^n\|^{1/n} = \lim_{n\to\infty} \frac{1}{|\lambda|} \|a\|^{1/n} < 1$. Therefore by Asymptotic bound, $\sum_{n=0}^{\infty} \lambda^{-1} a^n$ converges.

Recall that if $(\lambda 1 - a)^{-1} = \frac{1}{\lambda}(1 - \lambda^{-1} a)^{-1}$ we have $R(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n$ thus $\lambda \in \rho(a)$. By comparing $\sigma(a)$ and $\rho(a)$ we see that $r(a) = sup\{|\lambda| : \lambda \in \sigma(a)\} \leq s(a)$. To show equality between s(a), r(a) we let $r > r(a), \Omega = \{z \in \mathbb{C} : |z| > r(a)\} \subset \rho(a)$. For each $f \in A'$, we have $f \circ R : \Omega \to \mathbb{C}$ is analytic and has the series expansion $f \circ R(z) = \sum_{n \in \mathbb{Z}} c_n z^n$, but for $|z| > \|a\|$ we know $R(z) = \sum_{n=0}^{\infty} z^{-n-1} a^n$. Comparing coefficients give $c_n = 0$ for $n \geq 0$ and $c_{-1-n} = f(a^n)$ for $n \geq 0$. By our choice of z, $\lim_{n\to\infty} f(a^n) z^{-n} = 0$. Then by Banach-Steinhaus, we see that for $z \in \Omega$, $(z^{-n}a^n)_{n\in\mathbb{N}}$ is uniformly bounded. Therefore there exists $C > 0$ with $\|a^n\| \leq C|z|^n$ for all $n \in \mathbb{N}$. Thus $s(a) \leq \lim_{n\to\infty} C^{1/n}|z| = |z|$. This works for each z with $|z| > r(a)$. Taking the infimmum over all such z yields $s(a) \leq r(a)$. Thus $s(a) = r(a)$.

□

1.0.2 Remark. It is important to observe that the identity $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ relates algebraic and topological quantities without assuming C^* algebra structure.

2 Consequences of the Theory of the Spectrum

We explore the consequences of spectral theory.

2.0.1 Theorem. *Let A be a Banach Algebra with unit 1, in which each element $a \neq 0$ is invertible, then $A \cong \mathbb{C}$ and dim = 1.*

Proof. Let $a \in A$. By the above theorem on the spectrum, the exists $\lambda \in \sigma(A)$ s.t. $\lambda 1 - a \notin G(A)$ but by assumption $\lambda 1 - a = 0$ thus $a = \lambda 1$

□

In C^* algebras, we find more direct relationships between the spectrum and the norm

2.0.2 Lemma. *Let A be a C^* algebra.*

(i) $\sigma(a^*) = \overline{\sigma(a)}$

(ii) if a is normal, $r(a) = \|a\|$

(iii) for $a \in A$, $\|a\| = \sqrt{r(a^*a)}$

Proof. i) We know that $a - \lambda 1$ is invertible in $\tilde{A}$ iff $(a^* - \lambda 1)^* = a^* - \overline{A}1$. This gives the relation. □

The rest of the proof is to be continued....