MATH 7320 Lecture Notes

Note-taker: Kumari Teena

November 8, 2022

Last time:

- Spectrum Properties.

Warm up: Let $\mathcal{H} = \mathbb{C}^n$, $\mathcal{A} = \mathcal{B}(\mathbb{C}^n)$, $n \geq 2$. Define an operator a on $\mathcal{H}$ by

$$a \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ 0 \end{pmatrix},$$

then

$$a^* \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{pmatrix}.$$

Fix $(z_1, z_2, \cdots, z_n) \in \mathcal{H}$. Then for every $(w_1, w_2, \cdots, w_n) \in \mathcal{H}$, we have

$$\begin{aligned}
\langle (w_1, w_2, \cdots, w_n), a^*(z_1, z_2, \cdots, z_n) \rangle &= \langle a(w_1, w_2, \cdots, w_n), (z_1, z_2, \cdots, z_n) \rangle \\
&= \langle (w_2, w_3 \cdots, w_n, 0), (z_1, z_2, \cdots, z_n) \rangle \\
&= w_2 \overline{z_1} + w_3 \overline{z_2} + \cdots + w_n \overline{z_{n-1}} \\
&= \langle (w_1, w_2, w_3, \cdots, w_n), (0, z_1, z_2, \cdots, z_{n-1}) \rangle.
\end{aligned}$$

Thus,

$$a^*(z_1, z_2, \cdots, z_n) = (0, z_1, z_2, \cdots, z_{n-1}).$$

So,

$$a^*a \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{pmatrix}.$$

This implies that

$$(a^*a)^2 \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{pmatrix} = a^*a \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

So, $(a^*a)^2 = a^*a$. Hence, a^*a is an orthogonal projection and by $a^*a \neq 0$, we get $1 = \|a^*a\| = \|a\|^2$. So, $\|a\| = 1$.
However, $a^n = 0$, and same $a^m = a^{n+(m-n)} = 0$, for any $m \geq n$, and hence $r(a) = 0$. So, $\sigma(a) \neq \emptyset$, we know $\sigma(a) = \{0\}$.
We see that the spectral radius and the norm can be very different. We revisit the proof of theorem from last time

1 Relation between spectral radius and norm in C^*-algebra

Lemma 1. *Let $\mathcal{A}$ be C^*- algebra, then*
(i) $\sigma(a^*) = \overline{\sigma(a)} = \{\overline{z} : z \in \sigma(a)\}$.
(ii) If a is normal, then $r(a) = \|a\|$.
(iii) For $a \in \mathcal{A}$, $\|a\| = \sqrt{r(a^*a)}$.

Proof. (i) We know that $a - \lambda 1$ is invertible in $\tilde{\mathcal{A}}$ if and only if $(a - \lambda 1)^* = a^* - \overline{\lambda}1$. This gives the claimed relationship.
(ii) If a is normal, then so is a^2, because $a^2(a^2)^* = a^2(a^*)^2 = (aa^*)^2 =$

$(a^*a)^2 = (a^2)^*a^2$. Hence,

$$\begin{aligned}\|a^2\|^2 &= \|(a)^2(a^2)^*\| \\ &= \|(aa^*)(aa^*)\| \\ &= \|aa^*(aa^*)^*\| \quad (\textit{due to Hermitian}) \\ &= \|(aa^*)^2\| = \|a^4\| \ \ (\textit{By properties of } C^* - \textit{algebra}).\end{aligned}$$

$$\implies \quad \|a^2\| = \|a\|^2.$$

Inductively, $\|a^{2^n}\| = \|a\|^{2^n}$, for each $n \in \mathbb{N}$. Thus,

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|.$$

(iii) This follows from (ii). Since aa^* and a^*a are normal, then

$$\begin{aligned} r(aa^*) &= \|aa^*\| = \|a\|^2 \\ \implies \|a\| &= (r(aa^*))^{1/2}.\end{aligned}$$

□

Next we investigate how the spectrum behaves under homomorphisms.

Lemma 2. *Let $f : \mathcal{A} \longrightarrow \mathcal{B}$, be a homomorphism between algebras with unit $f(1) = 1$, then for $a \in \mathcal{A}$,*

$$\sigma(f(a)) \subset \sigma(a),$$

where $\sigma(f(a))$ is spectrum in $\mathcal{B}$ and $\sigma(a)$ is spectrum in $\mathcal{A}$.

Proof. We show equivalently that, $\rho(a) \subset \rho(f(a))$ (by taking compliments of set of spectrum). Here $\rho(a)$ denotes resolvant set of operator a. Let $\lambda \in \mathbb{C}$ be such that $a - \lambda 1 \in \mathcal{G}(\mathcal{A})$, with $\mathcal{R} = (a - \lambda 1)^{-1} \in \mathcal{A}$.
Applying f to $(a - \lambda 1)\mathcal{R}$ gives

$$\begin{aligned} f(a - \lambda 1)f(\mathcal{R}) &= (f(a) - \lambda 1)f(\mathcal{R}) \\ &= f(1) = 1.\end{aligned}$$

So, $f(a) - \lambda 1$ has right inverse $f(\mathcal{R})$. Similarly, $R(a - \lambda 1) = 1$ gives that $f(a) - \lambda 1$ has a left inverse.
So, $f(a) - \lambda 1$ is invertible with $(f(a) - \lambda 1)^{-1} = f(\mathcal{R}) \in \mathcal{B}$. This shows $\lambda \in \rho(f(a))$. □

Remark 3. *In particular, if $\mathcal{A} \subset \mathcal{B}$, $f = i_d$, then $\sigma(a)$ can only shrink when enlarging the algebra.*

Next, we study what happens when f respects the involution.

Theorem 4. *Let $\mathcal{A}$ be a Banach-$*$-algebra and $\mathcal{B}$ a C^*-algebra and $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism (f is algebra homomorphism, f is bounded, $f(a^*) = (f(a))^*$ for each $a \in \mathcal{A}$). Then, f is a contraction, i.e. $\|f(a)\| \leq \|a\|$, for all $a \in \mathcal{A}$.*

Proof. If we take $\mathcal{B} = \overline{f(\mathcal{A})}$, then the statement about $\|f\|$ is unaffected. So we can assume $f(\mathcal{A})$ is dense in $\mathcal{B}$.
If 1 is a unit in $\mathcal{A}$, then $f(1)f(a) = f(a)$. So, by C^*-algebra structure, $f(1)$ is a unit in $\mathcal{B}$. If $\mathcal{A}$ does not have a unit, then we extend $\mathcal{A}$, $\mathcal{B}$, and f to $\tilde{f} : \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{B}}$, $(a, \lambda) \longrightarrow (f(a), \lambda)$. Thus, $\tilde{f}(0, 1) = (0, 1)$. Now applying the preceding Lemma gives $\sigma(f(a)) \subset \sigma(a)$.
Consider, $a = a^* \in \mathcal{A}$, then by assumption on f, $f(a) = (f(a))^*$ and hence (since Hermitian are normal i.e. a is normal, $\|a\| = r(a)$, and $\|f(a)\| = r(f(a))$). So, $f(a)$ is normal.

$$\|f(a)\| = r(f(a)) \leq r(a) \leq \|a\|.$$

For general $a \in \mathcal{A}$, we have

$$\begin{aligned}
\|f(a)\|^2 &= \|f(a)^* f(a)\| \\
&= \|f(a^* a)\| \\
&\leq \|a^* a\| \\
&\leq \|a^*\| \|a\| \\
\implies \|f(a)\|^2 &\leq \|a\|^2.
\end{aligned}$$

□