Lecture Notes from November 11, 2022

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Last time Properties of the spectrum
Warm up:

1.47 Question. Relation between spectral radius and C^*-algebra norms.

Let $\mathcal{H} = \mathbb{C}^n$, $\mathcal{A} = \mathbb{B}(\mathbb{C}^n)$ $n \geq 2$.

$$a \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \cdot \\ z_n \\ 0 \end{pmatrix},$$

then for basis elements $e_1, e_2, \cdots e_n$

$$ae_1 = a \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \; ae_2 = a \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ 0 \\ 0 \end{pmatrix} = e_1 \text{ and } ae_k = e_{k-1}.$$

Hence, $\langle ae_k, e_j \rangle = \langle e_{k-1}, ej \rangle = \langle e_k, e_{j+1} \rangle = \langle e_k, a^* e_j \rangle$ and so $a^* e_k = e_{k+1}$.

$$a^* \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ z_1 \\ z_2 \\ \cdot \\ z_{n-1} \end{pmatrix}$$

Also,

$$(a^* a) \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ z_2 \\ z_3 \\ \cdot \\ z_n \end{pmatrix}$$

hence $(a^* a)$ is an orthogonal projection and since $(a^* a) \neq 0$ $\|a^* a\| = 1 = \|a\|^2$ so $\|a\| = 1$. (Observe that this is why we need $n \geq 2$.) However, $a^n = 0$ and so $a^m = a^{n+m-n} = 0$ for all $m \geq n$, hence $r(a) = 0$ but since $\sigma(a) \neq \phi$, we have $\sigma(a) \neq \{0\}$.

In areas such as numerical analysis, $\sigma(a^*a)$ is studied not $\sigma(a)$ to get more information as it is non-empty.

We revisit the lemma from last time.

1.48 Lemma. *Let $\mathcal{A}$ be a C^*-algebra. Then*

- $\sigma(a^*) = \overline{\sigma(a)} = \{\bar{\lambda} : \lambda \in \sigma(a)\}$.
- *If a is normal,* $r(a) = \|a\|$.
- *For $a \in \mathcal{A}$,* $\|a\| = \sqrt{r(a^*a)}$

Proof.

1. We know that $a - \lambda 1$ is invertible in $\tilde{A} \iff (a - \lambda 1)^* = a^* - \bar{\lambda} 1$ is invertible, i.e., $\overline{\lambda} \in (\sigma(a^*))^C \iff \lambda \in (\sigma(a))^C$.
2. If a is normal, $a^*a = aa^*$ then $a^2(a^2)^* = a\underline{aa^*}a^* = \underline{aa^*}\ \underline{aa^*} = a^*\underline{aa^*}a = a^*a^*aa = (a^*)^2a^2$, so a^2 is also normal. Hence,

$$\|a^2\|^2 \underbrace{=}_{\mathrm{C}^*\,alg.} \|a^2(a^2)^*\| = \|\|aa^*aa^*\|$$
$$= \|aa^*(aa^*)^*\|$$
$$\underbrace{=}_{\mathrm{C}^*\,alg.} \|aa^*\|^2$$
$$= \|a\|^4$$

   Taking square roots, we get $\|a^2\| = \|a\|^2$, and inductively we have $\|a^{2^n} = \|a\|^{2^n}$ for all $n \in \mathbb{N}$. Thus, considering a subsequence $(a^{2^n}) of (a^n)$ we get

$$r(a) = \lim_n \|a^n\|^{1/n} = \lim_n \|a^{2^n}\|^{1/2^n} = \|a\|$$

3. This follows from 2. since a^*a (and aa^*) is normal and so

$$r(a^*a) = \|a^*a\| = \|a\|^2$$

   and thus $\|a\| = \sqrt{r(a^*a)} = \sqrt{r(aa^*)}$.

□

Next, we investigate how the spectrum behaves under homomorphisms.

1.49 Lemma. *Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphisms between algebras woth unit s.t.* $f(1_{\mathcal{A}}) = 1_{\mathcal{B}}$*, then for any $a \in \mathcal{A}$,*

$$\sigma(f(a)) \subset \sigma(a)$$

Note that the left side is a spectrum in $\mathcal{B}$ and the right side is a spectrum in $\mathcal{A}$ (i.e., we consider inverses in those algebras respectively).

Proof. We show equivalently that $\rho(a) \subset \rho(f(a))$ (taking complements). Let $\lambda \in \mathbb{C}$ be such that $a - \lambda 1 \in G(\mathcal{A})$, with $R = (a - \lambda 1)^{-1} \in \mathcal{A}$. Applying f to

$$(a - \lambda 1)R \text{ gives } f(a - \lambda 1)f(R) = (f(a) - \lambda f(1))f(R) = (f(a) - \lambda)f(R).$$

Since $(a - \lambda 1)R = 1 \implies (f(a) - \lambda)f(R) = f(1) = 1$. Thus $f(R)$ is a right inverse of $f(a - \lambda 1)$. Similarly $(a - \lambda 1)R \implies f(R)(f(a) - \lambda) = f(1) = 1$. Thus $f(a - \lambda 1)^{-1} = f(R) \in \mathcal{B}$ and $\lambda \in \rho(f(a))$. In particular, if $\mathcal{A} \subset \mathcal{B}$, $f = \mathrm{id}$ then $\sigma(a)$ can only shrink when enlarging the algebra. □

Next, we study what happens when f respects the involution.

1.50 Theorem. *Let $\mathcal{A}$ be a Banach-$*$-algebra, $\mathcal{B}$ be a C^*-algebra, and $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism, i.e., it is a algebra homomorphism, bounded and respects the involution, then f is a contraction ($\|f(a)\| \leq \|a\|, \ \forall a \in \mathcal{A}$).*

Proof. When restricting $\mathcal{B}$ to be the closure of the range of f, then the statement on $\|f(a)\| \leq \|a\|$ is unchanged, so we can assume WLOG $f(A)$ dense in B. If 1 is a unit in $\mathcal{A}$, then $f(1)f(a) = f(a)$ so $f(1)$ is the id on $f(\mathcal{A})$ and by density of $f(\mathcal{A})$ and continuity of the product $f(1)b = b, \ \forall b \in \mathcal{B}$. By the C^*-algebra structure of $\mathcal{B}$, (left=right identity) $f(1)$ is a unit in $\mathcal{B}$. If $\mathcal{A}$ does not have a unit, $f(\mathcal{A})$ does not have a unit, so we extend $\mathcal{A}$, $\mathcal{B}$, and f:

$$\tilde{f} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$$

$$(a, \lambda) \mapsto (f(a), \lambda)$$

and thus $\tilde{f}(0, 1) = (0, 1)$. Now applying the preceeding lemma gives $\sigma(f(a)) \subset \sigma(a)$. Consider $a = a^* \in \mathcal{A}$, then $f(a^*) = f(a)^*$, $f(a) = f(a)^*$ implies $f(a)$ hermitian in $f(\mathcal{A})$. Since $f(a)$ is normal, $\|f(a)\| = r(f(a)) \leq r(a) \leq \|a\|$. For general, $a \in \mathcal{A}$, we have

$$\|f(a)\|^2 \underbrace{=}_{C^* alg.} \|aa^*\|^2 \|f(a)^* f(a)\| = \|f(a^* a)\| \leq \|a^* a\| \leq \|a^*\| \|a\| = \|a\|^2$$

since f is a contraction and $\|a^*\| = \|a\|$. □