Lecture Notes from November 10, 2022

taken by Dipanwita Bose

Last Time

- Spectral radius is norm in C^*- algebra.
- Properties of the spectrum under homomorphism, with or without respecting involution.

Finishing the theorem from last time-

2.51 Theorem. *Let A be a Banach $*$-algebra and B a C^*-algebra and $f : A \to B$ is a homomorphism(f is an algebraic homomorphism, is bounded and $f(a^*) = (f(A))^*$ for each $a \in A$). then f is a contraction. i.e., $\|f(a)\| \leq \|a\|$ for all $a \in A$.*

Proof. WLOG, let us suppose $B = \overline{f(A)}$, we observed $f(1) = 1$, or if A does not have unit we apply this to $\tilde{A}, \tilde{B}$ and achieve f extends to $\tilde{f}$ with $f(1) = 1$. Consider $a = a^* \in A$, then $f(a)$ is normal(since $f(a)(F(a))^* = f(a)f(a^*) = f(aa*) = f(a^*a) = f(a^*)f(a) = (f(a))^*f(a)$ as f is homomorphism). Now using spectral radius properties, we have

$$\|f(a)\| = r(f(a)) \overset{Lemma}{\leq} r(a) \leq \|a\|$$

For genral $a \in A$, we have

$$\begin{aligned}
\|f(a)\|^2 &= \|f(a)^* f(a)\| \\
&= \|f(a^* a)\| \\
&\leq \|a^* a\| \\
&\leq \|a^*\| \|a\| \\
&= \|a\|^2
\end{aligned}$$

using subadditivity of norm. Therefore we have $\|f(a)\| \leq \|a\|$ for any $a \in A$. □

Warm up: Without completeness of A and boundedness of f, the conclusion about the set of homomorphism does not hold. Consider the Banach-$*$-algebra $l^1(\mathbb{N})$ and $A = C_{00}$(Sequence with finitely many non-zero elements) in $l^1(\mathbb{N})$, then C_{00} is a subalgebra of $l^1(\mathbb{N})$ with norm and involution(identity).

For each $z \in \mathbb{C}$, we can define $f : A \to \mathbb{C}$ by $x \mapsto \sum_{n=1}^{\infty} x_n z^n$(where the sum is well defined as only finitely any x_n's are non-zero).
By A dense in $l^1(\mathbb{N})$ and $(l^1(\mathbb{N}))' = l^\infty(\mathbb{N})$, we have

$$\begin{aligned}
\|f\|_1 &= \sup_{\|x\|\leq 1}|f(x)| \\
&= \sup_{\|x\|\leq 1} abs \sum_{n=1}^{\infty} x_n z^n \\
&\leq \sup_{\|x\|\leq 1} \sum_{n=1}^{\infty} |x_n||z|^n \\
&\leq \sup_{\|x\|\leq 1} \|x\|_1 \sup_{n\in\mathbb{N}} |z|^n \\
&= \sup_{n\in\mathbb{N}} |z|^n
\end{aligned}$$

Moreover for $x = e_1$, a canonical basis element of $l^1(\mathbb{N})$ and $|z| < 1$,

$$|f(e_1)| = |z| = \sup_{n\in\mathbb{N}} |z|^n$$

Hence, we have $\|f\| = \sup_{n\in\mathbb{N}} |z|^n$. Also if $|z| > 1$, then $\|f\| = \infty$.

Note. We have a homomorphism which is everywhere defined but is discontinuous.

If $B = \mathbb{C}$, we can make a similar statement as in above theorem, but we do not require the presence of involution.

2.52 Theorem. *Let A be a Banach algebra, $f : A \to \mathbb{C}$ a homomorphism(i.e., in A'), then $\|f\| \leq 1$. If A has a unit and $f \neq 0$, then $f(1) = 1$, so $\|f\| = 1$.*

Proof. Let us assume $\|f\| > 1$, then there is $a \in A$ with $\|a\| = 1$ and $|f(a)| > 1$(using the definition of the operator norm), so by scaling we also get $\|a\| < 1$ and $f(a) = 1$(as $\|\frac{a}{|f(a)|}\| = \frac{\|a\|}{|f(a)|} < 1$ and $f(\frac{a}{|f(a)|}) = 1$. Now A being a Banach algebra, so by complexity of A,

$$b = \sum_{n=1}^{\infty} a^n \in A$$

where RHS converges(in norm) as $\|a\| < 1$. And by examining of the power series, we see

$$a + ab = a + a(\sum_{n=1}^{\infty} a^n) = a + \sum_{n=2}^{\infty} a^n = b$$

Applying f gives,

$$f(b) = f(a) + f(a)f(b) = 1 + f(b)$$

this is a contraction, so $\|f\| \leq 1$.

If A has a unit 1 and $f \neq 0$, then $f(1)^2 = f(1)$, so $f(1) = 1$ otherwise $f = 0$. Thus, $\|f\| = 1$, because $\|1\| = 1$. □

2.53 Definition. For a complex algebra A, we write Γ_A for the space of all non-zero homomorphism $\chi : A \to \mathbb{C}$(If A is Banach algebra, these elements of Γ_A are continuous, hence contractive).

We then cal Γ_A the spectrum of A.

2.54 Remarks. 1. If S is an involutive semigroup and $l^1(S)$ the Banach-$*$-algebra defined with $f^*(s) = f(s^*)$,

$$(f * g)(s) = \sum_{a,b \in S, ab=s} f(a)g(b),$$

then we had shown that continuous characters $\chi : l^1(S) \to \mathbb{C}$ come from bounded character on S by

$$\chi(f) = \sum_{s \in S} f(s)\gamma(s)$$

with $\gamma(s) = \chi(\delta_s)$, Now we know that all of these are infact contractive characters.

2. Let X be locally compact but not compact but not compact, $A = C_0(X)$. We show that each non-zero homomorphism $\chi : A \to \mathbb{C}$ is given by $\chi(f) = f(x)$ for $x \in X$.
   To see this, we extend χ to $\tilde{A} \subset C_b(X)$, by $\chi(1) = 1$.
   Let $\mathcal{N} = \ker(\chi)$, then $\mathcal{N}$ is a (closed) subspace of A of co-dimension one(by Rank-Nullity) and $\tilde{A}\mathcal{N} \subset \mathcal{N}$(since for $\tilde{a} \in \tilde{A}$, $x \in \ker(\chi)$, we have $\chi(\tilde{a}x) = \chi(\tilde{a})\chi(x) = \chi(\tilde{a}).0 = 0$)