Lecture Notes from November 10, 2022

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Last time

- Spectral radius vs norm in C^*-algebra.
- Properties of the spectrum under homomorphism, with or without respecting involution.

2.2 Theorem. *Let A be a Banach-*-algebra , B a C^*-algebra and $f : A \to B$ a homomorphism (bounded, algebra norm, *-preserving), then f is a contraction, i.e $\|f(a)\| \le \|a\|$ for each $a \in A$*

Proof. WOLG $B = f(A)$, we observed $f(\mathbb{1}) = \mathbb{1}$, or if A does not have a unit, we apply this to $\tilde{A}, \tilde{B}$ and achieve f extends to $\tilde{f}$ with $\tilde{f}(\mathbb{1}) = \mathbb{1}$.

Consider $b = b^* \in A$, then $f(b)$ is normal

$$\|f(b)\| = r(f(b)) \overset{\text{lemma}}{\le} r(b) \le \|b\|$$

For general $a \in A$, we have

$$\| \underbrace{f(a)}_{\in\ C^*\text{-algebra}} \|^2 = \|f(a)^*\ f(a)\| = \|f(\underbrace{a^*\,a}_{\text{Hermitian}})\| \overset{\text{replacing } b=a^*a}{\le} \|a^*\,a\| \le \|a^*\|\,\|a\| = \|a\|^2$$

□

2.3 Remark. Without completeness of A and boundedness of f, the conclusion about the set of homomorphisms does not hold.

2.4 Example. Consider the Banach-*-algebra $\ell^1(\mathbb{N})$ and $A = c_{00}$ (sequence with finitely many nonzero elements in $\ell^1(\mathbb{N})$), then c_{00} is a subalgebra of $\ell^1(\mathbb{N})$ with norm and involution (identity).

For each $z \in \mathbb{C}$, we can define

$$f : A \to \mathbb{C}$$
$$x \mapsto \sum_{n=1}^{\infty} x_n\, z^n$$

and by A is dense in $\ell^1(\mathbb{N})$ and $(\ell^1(\mathbb{N}))' = \ell^\infty(\mathbb{N})$, we have $\|f\| = \sup_{n\in\mathbb{N}} |z|^n$.

So if $|z| > 1$, then $\|f\| = \infty$.

2.5 Remark. If $B = \mathbb{C}$, we can make a similar statement as in the above statement, but we do not require the presence of involution.

2.6 Theorem. *Let A be a Banach algebra, $f : A \to \mathbb{C}$ a homomorphism (in A') then $\|f\| \leq 1$. If A has a unit and $f \neq 0$ then $f(\mathbb{1}) = 1$, so $\|f\| = 1$*

Proof. Let us assume $\|f\| \geq 1$, then there is $a \in A$ with $\|a\| = 1$ and $|f(a)| > 1$. So by scaling, let $\alpha = \frac{a}{f(a)}$, we also get $\|\alpha\| = \frac{\|a\|}{|f(a)|} < 1$ and $f(\alpha) = \frac{f(\alpha)}{f(\alpha)} = 1$.

For this α, by completeness of A, $b = \sum_{n=1}^{\infty} \alpha^n \in A$ where the RHS converges (in norm), and by examing the power series, we see

$$\alpha + \alpha b = \alpha + \alpha \sum_{n=1}^{\infty} \alpha^n = \sum_{n=1}^{\infty} \alpha^n = b$$

Applying f gives $f(b) = f(\alpha) + f(\alpha) f(b) = 1 + f(b)$. This is a contradiction, so $\|f\| \leq 1$.

If A has a unit $\mathbb{1}$ and $f \neq 0$, then $f(\mathbb{1})^2 = f(\mathbb{1}) f(\mathbb{1}) = f(\mathbb{1} \cdot \mathbb{1}) = f(\mathbb{1})$ so $f(\mathbb{1}) = 1$, otherwise $f(\mathbb{1}) = 0$. Thus $\|f\| = 1$ because $\|\mathbb{1}\| = 1$ □

2.7 Definition. For a complex algebra A, we write Γ_A for the space of all nonzero homomorphisms

$$\chi : A \to \mathbb{C}$$

If A is a Banach algebra, then elements of Γ_A are continuous, hence contractive. We then call Γ_A the spectrum of A

2.8 Remark.

(a) If S is an involutive semigroup and $\ell^1(S)$ the Banach-*-algebra defined with

$$f(s^*) = \overline{f(s^*)} \qquad \text{and} \quad (f * g)(s) = \sum_{\substack{a,b \in S \\ ab = s}} f(a) g(b)$$

then we had shown that the *continuous* chracters $\chi : \ell^1(S) \to \mathbb{C}$ come from bounded characters on S by $\chi(f) = \sum_{s \in S} f(s)\gamma(s)$ with $\gamma(s) = \chi(\delta_s)$.

Now we know that all of these are in fact contractive characters.

(b) Let X be locally compact but not compact $A = C_0(X)$. We have shown that each nonzero homomorphism $\chi : A \to \mathbb{C}$ is given by $\chi(f) = f(x)$ for $x \in X$.

To see this, we extend χ to $\tilde{A} \subset C_b(X)$ with $\chi(\mathbb{1}) = 1$. Let $\mathcal{N} = ker(\chi)$, then $\mathcal{N}$ is a closed subspace of $\tilde{A}$ of co-dimension one (by rank-nullity) and $\tilde{A}\mathcal{N} \subset \mathcal{N}$