Functional Analysis II, Math 7321
Lecture Notes from January 24, 2017

taken by Robert P Mendez

0 Course Information

Text: W. Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991 (or later).

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Grade: Based on preparation of class notes in LaTeX, rotating note-takers

Background knowledge: Linear algebra, Real analysis, Lebesgue integration

1 Previous Material

We begin the semester with a brief summary of some of the tools we used or developed last semester. To facilitate further reading, associated dates for class notes are given in square brackets; for additional convenience, each such reference is also a hyperlink to (one of) the class notes for that day.

1.1 Definition (Topological vector space [20 September 2016])**.** A vector space X together with a topology τ is called a *topological vector space* if

1. for every point $x \in X$, the singleton $\{x\}$ is a closed set, and
2. the vector space operations

$$\begin{aligned} + : X \times X &\to X \\ (x, y) &\mapsto x + y \end{aligned} \qquad \text{and} \qquad \begin{aligned} \cdot : \mathbb{K} \times X &\to X \\ (\lambda, x) &\mapsto \lambda x \end{aligned}$$

   are continuous with respect to the product topology on $X \times X$ and $\mathbb{K} \times X$, respectively.

An important consequence of this definition is that for $a \in X$, $T_a : A \to X$, $x \mapsto x + a$ is a homeomorphism, and so is, for $\lambda \neq 0$, $M_\lambda : x \mapsto \lambda x$. By the translation invariance of openess, the toplology is characterized by a filter of neighborhoods[1] of 0.

[1]NB: In the context of the class, we take the neighborhood of a point to be *any* set containing an *open* neighborhood of the point; Rudin, in our main text, uses "neighborhood" to mean "open neighborhood". Keep this in mind when interpreting proofs given in the text.

1.2 Definition (Types of topological vector spaces [20 September 2016]). We recall that there are several characteristics that topological vector spaces can have. A topological vector space is called

(a) *locally convex* if it has a local base of convex sets.

(b) *locally bounded* if 0 has a bounded neighborhood.

(c) *locally compact* if 0 has a compact neighborhood.

(d) *metrizable* if the topology is induced by a metric.

(e) an F-*space* if it is complete and the topology is induced by an invariant metric.

(f) a *Fréchet space* if X is a locally convex F-space.

(g) *normable* if the topology on X comes from a norm.

1.3 Characterization (Normable topological vector spaces [13 October 2016]). A topological vector space is normable if and only if it is locally bounded and locally convex.

1.4 Characterization (Locally compact topological vector spaces [4 October 2016]). A topological vector space X is locally compact if and only if it has finite dimension.

1.5 Remark (Motivation). We note that in cases when it is difficult to prove certain boundedness, convergence or convexity properties directly for a topological vector space X of interest, considering the underlying set with a coarser topology may provide insight. We illustrate this by recalling a special case of the weak topology[2] [Defined 17 November 2016]: Let $(X, \|\cdot\|)$ be a normed vector space and X' be the set of all continuous linear functionals on X. Defining X_w to be the set X with the initial topology τ_w induced by all elements in X', we call τ_w the weak topology on X.

If $\dim X = \infty$, then each $U \in \mathcal{U}(0)$ contains a nontrivial infinite dimensional subspace [Proposition $4.2.11$, 17 November 2016], from which we discern that the weak topology τ_w on X is *not* a locally bounded topological vector space. As a result, we have

1.6 Corollary. *If $(X, \|\cdot\|)$ has that $\dim X = \infty$, then $X \neq X_w$.*

We also recall the weak-$*$ topology, which is the initial topology induced by $\iota(X)$ on X' by

$$\iota(x) : f \mapsto f(x) \text{ for all } x \in X, f \in X',$$

and that the dual of X', equipped with the weak-$*$ topology is again X. This leads us to a (brief) study in the role of ι for normed spaces.

[2] This definition of the weak topology for a *normed* vector space follows from the [10 November 2016] result that in a locally convex topological vector space X, the dual space X^* of continuous linear functionals separates points in X.

2 Duality in Banach spaces

We first investigate the relationship between a normed space $(X, \|\cdot\|)$ and $(X', \|\cdot\|)$[3], where for $f \in X'$,

$$\|f\| := \sup_{\|x\| \leq 1} |f(x)|.$$

In analogy with Hilbert spaces, we write for $x \in X$ and $f \in X'$, $\langle f, x \rangle \equiv f(x)$. We might think of this apparent abuse of inner product notation as emphasizing the linearity of f; we may further justify its use by noting that the definition of the respective norms yields $|\langle f, x \rangle| \leq \|f\|\|x\|$, reminiscent of the Cauchy-Schwarz inequality.

2.1 Definition. Let X be a normed space. The mapping $\iota : X \to X'' \equiv (X')'$ defined $\langle \iota(x), f \rangle := \langle f, x \rangle$ is called the *natural inclusion* of X' in X''. Since continuity is equivalent to boundedness in linear maps, we have the following propostion:

2.2 Proposition. *The natural inclusion is a linear, norm-preserving map.*

Proof. We note that for each $f \in X'$,

$$|\langle \iota(x), f \rangle| = |\langle f, x \rangle| \leq \|f\|\|x\|,$$

and it follows that $\|\iota(x)\| \leq \|x\|$. On the other hand, applying Hahn-Banach,

$$\|x\| = \max_{\|f\| \leq 1} |\langle f, x \rangle| = \max_{\|f\| \leq 1} |\langle \iota(x), f \rangle| \leq \|\iota(x)\| \max_{\|f\| \leq 1} \|f\| = \|\iota(x)\|$$

yields that $\|\iota(x)\| \geq \|x\|$, so the norm is preserved. □

2.3 Remark. By this isometry property, ι is injective, so it embeds X in X''. However, ι is not necessarily an isometric isomorphism; we give *that* case a special name:

2.4 Definition. A Banach space X is called reflexive if the natural inclusion ι of X in X'' is an isometric isomorphism.

2.5 Remark (1). We note that the definition describes a Banach space, and not the more general *normed* space. Since the linear functionals in X' map into the complete space $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the dual space inherits completeness. It follows that the dual of X' is complete, and so a normed vector space that is reflexive is therefore complete.

2.6 Remark (2). It is not enough to have that X and X'' are isomorphic–reflexivity is a property of ι!

We proceed with some easy results in reflexivity.

2.7 Proposition. *Every finite dimensional Banach space is reflexive.*

2.8 Remark. Since a finite dimensional Banach space is essentially $\mathbb{R}^n$ or $\mathbb{C}^n$, so is its dual, and then so is its "double dual".

[3]Technically, the norms are $\|\cdot\|_X$ and $\|\cdot\|_{X'}$. We suppress the subscript with the understanding that context should make clear which norm is implied.

Proof. Let $\dim X = n \in \mathbb{N}$. By biorthogonal basis pairs, $\dim X' = n$, and so follows that $\dim X'' = n$. Since ι is norm preserving, it is injective; it follows from rank-nullity, then, that ι is surjective, as well. □

We prepare for the next result by recalling a Riesz representation theorem.[4] For clarity, since we are already using $\langle \cdot, \cdot \rangle$ to indicate functional evaluation, we shall use $(\cdot, \cdot)_X$ to indicate the inner product on the Hilbert space X:

2.9 Theorem (F. Riesz, [Theorem 4.8, **?**]). *Given Hilbert space H, every $x \in H$ induces a continuous linear functional on H by $\varphi_x(y) = (x, y)_H$; additionally, we have that $\|\varphi_x\| = \|x\|$ for all $x \in H$, and that this mapping of H onto H' is bijective and antilinear, so that*

$$\varphi_{ax+by} = a^* \varphi_x + b^* \varphi_y.$$

We note that the antilinearity implies that $(x, y)_H = (\varphi_y, \varphi_x)_{H'}$.

2.10 Proposition. *Every Hilbert space H is reflexive.*

Proof. Applying the Riesz representation theorem[5] above, let Φ and Θ give the maps

$$\begin{aligned} \Phi : H &\to H' \\ x \mapsto \varphi_x :& H \to \mathbb{K} \\ & y \mapsto (x, y)_H \equiv \langle \varphi_x, y \rangle \end{aligned}$$

and

$$\begin{aligned} \Theta : H' &\to H'' \\ \varphi \mapsto f_\varphi :& H' \to \mathbb{K} \\ & \vartheta \mapsto (f_\varphi, \vartheta)_H \equiv \langle f_\varphi, \vartheta \rangle . \end{aligned}$$

We claim that $\iota := \Theta \circ \Phi$ is the isometric isomorphism we desire.

First, ι inherits the norm preservation from the composition. Linearity follows from the fact[6] that for $a, b \in \mathbb{K}$ and $x, y \in H$,

$$f_{\varphi_{ax+by}} = f_{a^* \varphi_x + b^* \varphi_y} = a f_{\varphi_x} + b f_{\varphi_y};$$

To show that $\iota : H \hookrightarrow H''$ is the natural inclusion map, we must show that $\langle \iota(x), \varphi_y \rangle = \langle \varphi_y, x \rangle$ for all $x, y \in H$. Rewriting the left-hand side, we have

$$\langle \iota(x), \varphi_y \rangle = \langle f_{\varphi_x}, \varphi_y \rangle = (\varphi_x, \varphi_y)_{H'} = (y, x)_H = \langle \varphi_y, x \rangle ,$$

completing the proof.

□

References

Weidmann, Joachim, *Linear Operations in Hilbert Spaces*, Springer-Verlag New York, 1980.

[4]There are a number of theorems attributed to Frigyes Riesz which are called "the Reisz representation theorem." Here, we are specifying which explicitly.

[5]Suggest further reading: Joachim Weidmann's *Linear Operators on Hilbert Spaces*, Springer-Verlag New York, 1979.

[6]Alternatively, we gain isomorphism from $(x, y)_H = (\varphi_y, \varphi_x)_{H'} = \left(f_{\varphi_x}, f_{\varphi_y}\right)_{H''}$