Functional Analysis II, Math 7321
Lecture Notes from January 26, 2017

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Recall that a topological space is separable if it has a countable dense set. In this note, we are going to characterize a separable Banach space. First, we give some examples of Banach spaces which is not separable.

2.11 Examples. l_∞, the set of bounded sequences with a norm $\|(x_n)\| = \sup_{i\in\mathbb{N}} |x_n|$, is a Banach space. But, it is not separable.

Proof. It is well known that l_p is Banach space for any $1 \le p \le \infty$. We only prove that l_∞ is not separable. Let $B = \{(x_n)_{n=1}^\infty : x_n \in \{-1,1\}\} \subseteq l_\infty$. Then, B is uncountable and $\|(x_n)-(y_n)\| = \sup_{n\in\mathbb{N}} |x_n - y_n| = 2$ for any distinct pair $(x_n),(y_n) \in B$. Thus, $B_1((x_n)) \cap B_1((y_n)) = \emptyset$ for any distinct $(x_n),(y_n) \in B$. Let D be a dense set in l_∞. Thus, for any $(x_n) \in B$, there is at least one element in D is in $B_1((x_n))$. Since each $B_1((x_n))$ is disjoint from others, D must be uncountable as same as B. This shows l_∞ is not separable. $\square$

More generally, L_∞, the space of bounded (a.e.) measurable functions, is not separable.

Next, we provide an example of reflexive Banach spaces. Recall that a Banach space X is reflective if the canonical embedding of X in X'' is an isometric isomorphism.

2.12 Examples. The space ℓ_p where $1 < p < \infty$ is reflexive. To show ℓ_p is reflexive, we first show ℓ_p' and ℓ_q are isometrically isomorphic if $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$. Let $x \in \ell_p$, then by Hölder's inequality, $|\langle y, x\rangle| = |\sum_{n\in\mathbb{N}} y_n x_n| \le \|y\|_q \|x\|_p$, where the series converges absolutely/unconditionally, so this defines a continuous linear functional $f_x : y \mapsto \langle y, x\rangle$ on ℓ_q. Moreover, equality is assumed in Hölder's inequality, so the map $x \mapsto f_x \equiv \langle \cdot, x\rangle$ is an isometry. Conversely, for each $f \in \ell_q'$, let y be the sequence with entries $y_n = f(e_n)$, $(e_n)_m = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{otherwise} \end{cases}$ then for each $x \in \ell_q$ by linearity and continuity, $f(x) = \sum_{n\in\mathbb{N}} y_n x_n$ and the series converges absolutely, which can be seen by taking $x_n|f(e_n)| = |x_n| f(e_n)$. We leave showing that $\|y\|_q < \infty$ as exercise.

Using the identification $\ell_p' \simeq \ell_q$ twice gives $\ell_p \simeq \ell_p''$. Moreover, by the above the map $i : x \mapsto \langle \cdot, x\rangle$ defines an isomorphism between ℓ_p and ℓ_p''.

2.A Quotient Space

2.13 Definition. Let V be a vector space and M a linear subspace. We say $x, y \in V$ are equivalent modulo M if $x - y \in M$. The set of all equivalent classes is called Quotient space

V/M and denote the equivalent class with representative $x \in V$ as

$$[x] = \{y \in X : y - x \in M\} = x + M.$$

2.14 Examples.

- Let $m, n \in \mathbb{N}$. Define $M = \{(x_1, ..., x_m, 0, ..., 0) : x_1, ..., x_m \in \mathbb{R}\}$. Then, M is a subspace of $\mathbb{R}^{m+n}$ and $R^{m+n}/M = \{[0, 0, ..., 0, y_1, ..., y_n] : y_1, ..., y_n \in \mathbb{R}\}$. Thus, $\mathbb{R}^{m+n}/M$ is isomorphic to $\mathbb{R}^n$.
- In general, if a vector space V is a direct sum ot subspaces M and N, i.e., $V = M \bigoplus N$. Then, V/M is isomorphic to N and V/N is isomorphic to M.

From the definition, by defining $[x] + [y] = [x + y]$ and $\lambda \cdot [x] = [\lambda x]$ for $x, y \in V$ and $\lambda \in \mathbb{R}$, with these two operations, the quotient space is also a vector space. In addition, if V is a normed space and M is closed linear subspace, then

$$\|x\|_{X/M} = \inf_{y \in M} \|x - y\|$$

defined a norm in V/M. Thus, V/M is also a normed and we state this fact as following.

2.15 Proposition. *Equipped with $\|\cdot\|_{X/M}$, the quotient space X/M becomes a normed space.*

Proof. Let $[x], [x'] \in X$ and $\lambda \in \mathbb{K}$. Then,

$$\|[\lambda x]\| = \inf_{y \in M} \|\lambda x - y\| = \inf_{y \in M} \|\lambda x - \lambda y\| = |\lambda| \inf_{y \in M} \|x - y\| = |\lambda| \|[x]\|.$$

Also, if $\|[x]\| = 0$, then $\inf_{y \in M} \|x - y\| = 0$. We have a sequence (y_n) in M converging to x. Since M is closed, $x \in M$. Thus, $[x] = M$. Finally, $\|[x + x']\| = \inf_{y \in M} \|x + x' - y\| = \inf_{y \in M} \|(x - y) + (x' - y)\| \leq \inf_{y \in M} (\|(x - y)\| + \|(x' - y)\|) \leq \inf_{y \in M} \|(x - y)\| + \inf_{y \in M} \|(x' - y)\|) = \|[x]\| + \|[x']\|$. Therefore, $\|\cdot\|$ is a norm. □

The next proposition was proved from the last semester. Thus, we will only sketch the proof of this proposition.

2.16 Proposition. *If B is a Banach space and M is a closed subspace of B, then B/M is a Banach space.*

Proof. Let $([x_n])$ be a Cauchy sequence in B/M. We can choose $y_n \in B$ which $y_n \in [x_n]$ and (y_n) is a Cauchy sequence in B. Thus, y_n converges to $y \in B$. Then, we can show that $[x_n]$ converges to $[y]$. Therefore, B/M is a Banach space. □

There is natural map $q : V \to V/M$ which maps an element in V to its equivalent class i.e., $q(x) = [x]$. We call q a quotient map. One interesting property of this map is that it maps the unit open ball on V onto the unit open ball on V/M.

2.17 Proposition. *If X is a normed space and M be a sub space, then $q : X \to X/M$ maps the (open) unit ball onto the unit ball in X/M.*

Proof. Let $x \in X$. We have

$$\|q(x)\|_{X/M} = \inf_{y\in[x]} \|y\|_X \le \|x\|.$$

So, q maps the ball in X into the one in X/M. Next, we want to show $q(B_1^X(0)) = B_1^{X/M}(0)$. Let $y \in B_1^{X/M}(0)$. Then, $1 > \|y\|_{X/M} = \inf_{x\in[y]} \|x\|_X$. So, there is $x \in X, m \in M$ such that $\|x+m\|_X < 1$. Hence, $q(x+m) = y$ and there is $x+m \in B_1^X(0)$ that maps to y. $\square$

From the previous proposition, the quotient map maps the unit open ball onto the unit open ball. Conversely, if we have a linear map $T : X \to Y$ which maps the unit open ball on a normed space X onto the unit ball on a normed space Y, then Y is isometrically isomorphic to $X/\ker T$ and thus T can be considered as a quotient map from $X \to Y$.

2.18 Proposition. *Let X, Y be normed spaces. Let $T : X \to Y$ be a linear map onto Y, and assume $B_1^X(0)$ is mapped onto $B_1^Y(0)$, then Y is isometrically isomorphic to $X/\ker T$.*

Proof. Let $[x] = \{x + y : y \in \ker T\}$ and define $q : X/\ker T \to Y$ by $q(x) = T(x)$. Then q is linear and surjective. Moreover, q is injective because $q([x]) = 0$ implies $x \in \ker T$ i.e., $[x] = \ker T = [0]$. Hence, q is bijection mapping open ball to the open ball. The same holds for q^{-1}. From $\|q\| \le 1$ and $\|q^{-1}\| \le 1$, $\|q\| = \|q^{-1}\| = 1$. Next, we have

$$\|x\|_{X/\ker T} = \|q^{-1}(q([x]))\|_{X/\ker T} \le \|q([x])\|_Y \le \|[x]\|_{X/\ker T}.$$

Thus, $\|q([x])\| = \|[x]\|$, i.e., q is isometric isomorphism. $\square$

2.19 Lemma. *Let B be a Banach space and A a dense set in $B_r(0)$, then every $x \in B_r(0)$ can be expressed as $x = \sum_{j=1}^\infty \lambda_j x_j$ where $x_j \in A$ and $\sum_{j=1}^\infty |\lambda_j| < 1$.*

Proof. Without loss of generality, let $r = 1$. Take $x \in B_1(0)$. Choose $\delta > 0$ such that $(1+\delta)\|x\| < 1$. Fix $\varepsilon > 0$ and let $x_1 \in A$ be such that $\|(1+\delta)x - x_1\| < \varepsilon$. From $\|\frac{(1+\delta)x-x_1)}{\varepsilon}\| < 1$, there is $x_2 \in A$ with

$$\left\|\frac{((1+\delta)x - x_1)}{\varepsilon} - x_2\right\| < \varepsilon.$$

Next, we take $x_3 \in A$ for which

$$\left\|\frac{(1+\delta)x - x_1 - \varepsilon x_2}{\varepsilon^2} - x_3\right\| < \varepsilon.$$

So,

$$\|(1+\delta)x - x_1 - \varepsilon x_2 - \varepsilon^2 x_3\| < \varepsilon^3.$$

Continuing inductively gives

$$x = \frac{1}{1+\delta}\sum_{j=1}^\infty \varepsilon^{j-1} x_j.$$

Setting $\lambda_j = \frac{\varepsilon^{j-1}}{1+\delta}$ gives $\sum_{j=1}^\infty |\lambda_j| = \frac{1}{(1+\delta)(1-\varepsilon)}$. Choose ε sufficeiently small, $\|\lambda\|_1 < 1$. $\square$

The following is our main goal in this lecture. We have proved important facts and will use them to characterize a separable Banach space.

2.20 Theorem. *Every separable Banach space B is isometrically isomorphic to a quotient space of l_1.*

Proof. Take (x_n) to be dense in $B_1(0)$. Let $T : l_1 \to B$ defined by $T(c) = \sum_{j=1}^{\infty} c_j x_j$ for $c \in l_1$. Then, by Minkowski's inequality, $\|T(c)\| \leq \sum_{j=1}^{\infty} |c_j| = \|c\|$. So, $\|T\| \leq 1$. By the preceding lemma, $T(B_1^{l_1}(0)) = B_1^B(0)$. So, B is isomorphic to $l_1 / \ker T$. $\square$

Next, we relate maps on Banach spaces to maps on their duals.

2.21 Definition. Let X, Y be normed spaces and $T \in B(X, Y)$, the set of all bounded linear maps. We define the adjoint map $T' : Y' \to X'$ as

$$< T'(f), x >=< f, T(x) >,$$

for each $x \in X$ and $f \in Y'$.

2.22 Remark. The definition above is well defined since for $f \in Y'$, $T'f \in X'$, and we obtain

$$| < f, T(x) > | \leq \|f\| \|Tx\| \leq \|f\| \|T\| \|x\|.$$