Functional Analysis, Math 7321
Lecture Notes from January 31, 2017

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Last time we defined the adjoint of a bounded, linear map between normed vector spaces. By definition, the adjoint of a map $T : X \to Y$ acts by "pulling back" linear functionals on Y to linear functionals on X. Below, we show that the adjoint is itself a bounded linear map from Y' to X', and that its norm is well-behaved.

2.23 Proposition. *If $T \in B(X, Y)$ where X and Y are normed spaces, then $T' \in B(X', Y')$, and $\|T'\| = \|T\|$.*

Proof. First, we show T' is in fact linear. Take $f, g \in Y'$ and $\alpha \in \mathbb{K}$. Then for any $x \in X$,

$$\begin{aligned}\langle T'(\alpha f + g), x\rangle &= \langle \alpha f + g, T(x)\rangle && \text{(by definition of the adjoint)}\\ &= \alpha\langle f, T(x)\rangle + \langle g, T(x)\rangle && \text{(by linearity of the dual pairing)}\\ &= \alpha\langle T'(f), x\rangle + \langle T'(g), x\rangle.\end{aligned}$$

Since this holds for all $x \in X$, we see $T'(\alpha f + g) = \alpha T'(f) + T'(g)$. Thus T' is linear.

To calculuate the norm of $\|T'\|$, we use that for $f \in Y'$,

$$|\langle T'(f), x\rangle| \overset{(1)}{=} |\langle f, T(x)\rangle| \overset{(2)}{=} |f(T(x))| \overset{(3)}{\leq} \|T\|\|f\|\|x\|,$$

where (1) follows from the definition of the adjoint, (2) by the definition of the dual pairing, and (3) by the fact that f and T are bounded. Taking the supremum on both sides over all $x \in X$ with $\|x\| \leq 1$ yields:

$$|T'(f)| = \sup_{x \in X, \|x\| \leq 1} |\langle T'(f), x\rangle| \leq \|T\|\|f\|.$$

Thus T' is bounded, so $T' \in B(Y', X')$, and $\|T'\| \leq \|T\|$.

To prove the reverse inequality, recall the following corollary to Hahn-Banach (discussed in class on 11/8/16): *If Z is a normed space and $b_0 \in Z$, there exists $f \in Z'$ such that $f(z_0) = \|z_0\|$ and $|f(z)| \leq \|z\|$ for all $z \in Z$.* For any $x \in X$:

$$\|T(x)\| \overset{(4)}{=} \max_{g \in Y', \|g\| \leq 1} |\langle g, T(x)\rangle| \overset{(5)}{=} \max_{g \in Y', \|g\| \leq 1} |\langle T'(g), x\rangle| \overset{(6)}{\leq} \|T'\|\|x\|,$$

where (4) follows from the aforementioned corollary, (5) by definition of the adjoint, and (6) by the fact that T' is bounded (shown above). Thus $\|T\| \leq \|T'\|$, so equality holds. □

If we have a bounded linear map $T : X \to Y$ between normed spaces, we can view $T' : Y' \to X'$ as a sort of "mirror image" of T, in that it reverses direction yet mimics the properties of T. In particular, we show that the adjoint of an invertible map is also invertible and determine its inverse.

2.24 Lemma. *If $T \in B(X, Y)$ is invertible, then $T' \in B(Y', X')$ is also invertible. Moreover, $(T')^{-1} = (T^{-1})'$.*

Proof. Since T is invertible we know $T^{-1} \in B(Y, X)$, so we may consider $(T^{-1})' \in B(X', Y')$. Let $S = (T^{-1})'$.

For any $f \in X'$ and any $x \in X$:

$$\langle T'(S(f)), x\rangle = \langle S(f), T(x)\rangle = \langle f, T^{-1}(T(x))\rangle = \langle f, x\rangle.$$

Thus $T'(S(f)) = f$ for all $f \in X'$, so S is a right-inverse for T'.

Similarly, for any $g \in Y'$ and any $y \in Y$:

$$\langle S(T'(g)), y\rangle = \langle T'(g), T^{-1}(y)\rangle = \langle g, T(T^{-1}(y))\rangle = \langle g, y\rangle.$$

Thus $S(T'(g)) = g$ for all $g \in Y'$, so S is a left-inverse for T'. This means T' is invertible and $(T')^{-1} = S = (T^{-1})'$. □

We can use the above lemma to show that T' inherits the isometry property from T.

2.25 Proposition. *If $T \in B(X, Y)$ is an (isometric) isomorphism, then so is $T' \in B(Y', X')$.*

Proof. By the above lemma we know T' is invertible, and $(T')^{-1} = S = (T^{-1})'$. It remains to show that T' is an isometry.

Since T is an isometric isomorphism we know $\|T\| = 1$ and $\|T^{-1}\| = 1$. The earlier proposition tells us that $\|T'\| = \|T\| = 1$, and also that $\|(T')^{-1}\| = \|(T^{-1})'\| = \|T^{-1}\| = 1$. So for any $g \in Y'$:

$$\|T'(g)\| \leq \|g\| = \|(T')^{-1}(T'(g))\| \leq \|(T')^{-1}\|\|T'(g)\| = \|T'(g)\|,$$

so equality holds throughout. Thus T' is an isometry.

□

The above proposition says more than just $X \cong Y \implies X' \cong Y'$. It gives a specific, canonical way to convert an isomorphism of Banach spaces into an isomorphism of their duals (i.e., take the inverse of the adjoint).

2.A Annihilators

Next we examine the geometric aspects of duality. Our definitions and notation are motivated from those in Hilbert spaces, highlighting the parallel between the bilinear pairing $\langle f, x\rangle$ (of x in a Banach space X and $f \in X'$) and the bilinear/sesquilinear inner product $\langle v, w\rangle$ (of elements v, w in a Hilbert space). The following definitions and propositions can be phrased in an even more general setting, the only requirement is that we have a locally convex topological vector space.

2.26 Definition. Let B be a Banach space, M a subspace of B, and N a subspace of B'. We write the *annihilators* of M and N as:

$$\begin{aligned} M^{\perp} &= \{f \in B' : \langle f, M\rangle = \{0\}\} \\ N^{\perp} &= \{x \in B : \langle N, x\rangle = \{0\}\}. \end{aligned}$$

We next explore some topological properties of annihilators.

2.27 Proposition. *Let B be a Banach space, M a subspace of B, and N a subspace of B'. Then $M^{\perp}$ and $N^{\perp}$ are closed subspaces.*

Proof. Recall that if X is a first-countable topological space and $S \subset X$, then $x \in \overline{S}$ iff there exists a sequence of elements $(s_n)_{n\in\mathbb{N}} \subset S$ such that $s_n \to x$. In other words, the sequential closure is the same as the topological closure. For this proposition B and B' are first-countable because they are Banach spaces (obviously metrizable), so to show $M^{\perp}$ and $N^{\perp}$ are closed we need only show that they are sequentially closed.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements in $M^{\perp}$ that converges to some $f \in B'$. By the continuity of $\langle\cdot,\cdot\rangle$ in the first argument, we see that for any $m \in M$, $\langle f_n, m\rangle$ converges to $\langle f, m\rangle$. But since $\langle f_n, m\rangle = 0$ for all $n \in \mathbb{N}$, we have that $\langle f, m\rangle = 0$. Since $m \in M$ was arbitrary we see $f \in M^{\perp}$, thus $M^{\perp}$ is closed.

Next, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in $N^{\perp}$ that converges to some $x \in B$. By the continuity of $\langle\cdot,\cdot\rangle$ in the second argument, we see that for any $n \in N$, $\langle n, x_n\rangle$ converges to $\langle n, x\rangle$. But since $\langle n, x_n\rangle = 0$ for all $n \in \mathbb{N}$, we have that $\langle n, x\rangle = 0$. Since $n \in N$ was arbitrary we see $x \in N^{\perp}$, thus $N^{\perp}$ is closed. □

2.28 Proposition. *If M is a subspace of a Banach space B, then $(M^{\perp})^{\perp} = \overline{M}$.*

Proof. Let $x \in M$, then for each $f \in M^{\perp}$, $\langle f, x\rangle = 0$ so $x \in (M^{\perp})^{\perp}$. Thus $M \subset (M^{\perp})^{\perp}$, and because $(M^{\perp})^{\perp}$ is closed by the above proposition we also see $\overline{M} \subset (M^{\perp})^{\perp}$.

To show the reverse inequality, we will use the following corollary of Hahn-Banach: *Suppose Y is a subspace of a locally convex space X, and $x_0 \in X$. If $x_0 \notin \overline{Y}$, then there exists $\Lambda \in X'$ such that $\Lambda(x_0) = 1$ but $\Lambda(x) = 0$ for all $x \in Y$.* Note: We can apply this corollary because every normed space is locally convex.

Let $x_0 \notin \overline{M}$. Then there is a linear functional $f \in B'$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in M$. So $f \in M^{\perp}$, but $\langle f, x_0\rangle = 1 \neq 0$, thus $x_0 \notin (M^{\perp})^{\perp}$. So we have $\overline{M}^c \subset ((M^{\perp})^{\perp})^c$, which implies $(M^{\perp})^{\perp} \subset \overline{M}$. □

We now formulate a generalization of the classical rank-nullity theorem.

2.29 Theorem. *Let X and Y be normed spaces and $T \in B(X,Y)$. Then $\ker T = (\text{ran } T')^{\perp}$ and $\ker T' = (\text{ran } T)^{\perp}$.*

Proof. We identify the sets as follows:

$$\begin{aligned} x \in \ker T &\iff Tx = 0 \\ &\iff \langle f, Tx\rangle = 0, \forall f \in Y' \\ &\iff \langle T'f, x\rangle = 0, \forall f \in Y' \\ &\iff x \in (T'(Y'))^{\perp} = (\text{ran } T')^{\perp}. \end{aligned}$$

Similarly, for the second identification:

$$\begin{aligned} f \in \ker T' &\iff T'f = 0 \\ &\iff \langle T'f, x\rangle = 0, \forall x \in X \\ &\iff \langle f, Tx\rangle, \forall x \in X \\ &\iff f \in (\text{ran } T)^{\perp}. \end{aligned}$$

□

2.30 Remark. Consider the above theorem in the context where $T : X \to Y$ is a linear map between finite-dimensional Hilbert spaces. In this case, by the Riesz representation theorem the adjoint map $T' \in B(Y', X')$ can be identified with a map $T^* : Y \to X$. If we let A be the matrix representation of T with respect to some fixed bases for X and Y, then the corresponding matrix representation for T^* is A^* (where here * denotes the conjugate transpose). Using the Hilbert space notion of orthogonality, note that $Ax = 0$ iff $\langle r, x\rangle = 0$ for each row r of A, i.e. iff $x \in (\text{row}(A))^{\perp} = (\text{col}(A^*))^{\perp}$. Thus $\ker(T) = (\text{ran}(T^*))^{\perp}$. Similarly, we also get that $\ker(T^*) = (\text{ran}(T))^{\perp}$.

We next study the interplay between annihilators and quotient spaces.

2.31 Proposition. *Let M be a closed subspace of a Banach space B, then:*

1. *M' is (isometrically) isomorphic to $B'/M^{\perp}$.*
2. *$(B/M)'$ is (isometrically) isomorphic to $M^{\perp}$.*

Proof. For (1), define a map $\sigma : M' \to B'/M^{\perp}$ by $\sigma(f) = [F]$, where $F \in B'$ is an extension of f (possible by Hahn-Banach). Note that the equivalence class $[F]$ is independent of the particular choice of the extension F, because if F and G are both extensions of f, then:

$$F|_M = G|_M \implies (F-G)|_M = 0 \implies F - G \in M^{\perp} \implies [F] = [G].$$

Also note that σ is linear by definition. It is also surjective, since if $[F] \in B'/M^{\perp}$ then $[F] = \sigma(F|_M)$.

It remains to show that σ is an isometry. By definition, $\|\sigma(f)\| = \|[F]\| = \inf_{G\in[F]} \|G\|$. Since any extension G of f must satisfy $\|G\| \geq \|f\|$, we have that $\inf_{G\in[F]} \|G\| \geq \|f\|$. Also, since Hahn-Banach gives an extension of f with the same norm as f, we get $\inf_{G\in[F]} \|G\| \leq \|f\|$, so equality holds. Thus σ is an isometry.

For (2), consider $\tau : (B/M)' \to B'$ defined by setting $\langle \tau(f), x\rangle = \langle f, [x]\rangle$. Note that if $x \in M$, then $[x] = [0]$ so:

$$\langle \tau(f), x\rangle = \langle f, [x]\rangle = \langle f, [0]\rangle = 0.$$

Thus $\tau(f) \in M^\perp$ for every $f \in (B/M)'$, so ran $(\tau) \subset M^\perp$. Also, if $g \in M^\perp$, then there is some $f \in (B/M)'$ with $\langle f, [x]\rangle = \langle g, x\rangle$, so $g = \tau(f)$. Thus τ is onto $M^\perp$.

We will conclude this proof next time by showing that τ is an isometry. □

Last time we saw that every separable Banach space is isometrically isomorphic to a quotient space of ℓ_1. Combining this observation with the above proposition yields the following characterization of the dual spaces of separable Banach spaces.

2.32 Corollary. *The dual of a separable Banach space is isometrically isomorphic to a closed subspace of ℓ_∞.*

Proof. Given a Banach space B there exists a closed subspace $W \subset \ell_1$ such that $B \cong \ell_1/W$. From item (2) above, we see that $(\ell_1/W)' \cong W^\perp$, so $B' \cong W^\perp$. In this case, $W^\perp$ is a closed subset of $\ell_1' = \ell_\infty$. □