Functional Analysis II, Math 7321
Lecture Notes from February 02, 2017

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Last Time: Annihilators, rank-nullity, quotient spaces and duality

The following proposition was previously stated during last lecture.

2.33 Proposition. *Let M be a closed subspace of a Banach space B. Then*

(i) M is isometrically isomorphic to $B'/M^\perp$.

(ii) $(B/M)'$ is isometrically isomorphic to $M^\perp$.

Proof. The first result was proved last time.

(ii) Consider $\tau : (B/M)' \to B'$, $\langle \tau(f), x\rangle = \langle f, [x]\rangle$. Note that if $x \in M$, then $[x] = [0]$, so $\langle \tau(f), x\rangle = \langle f, [0]\rangle = 0$. Thus, Range($\tau$) $\subset M^\perp \subset B'$. Next, one must show surjectivity of τ on $M^\perp$. If $y \in M^\perp$, then there is a $f \in (B/M)'$ with $\langle f, [x]\rangle = \langle g, x\rangle$. Hence, $g = \tau(f)$. Finally, one shows τ preserves the norm. Let $f \in (B/M)^\perp$, $[x]$ in the unit sphere of B/M and $\epsilon > 0$. Then there is $y \in [x]$ with $1 \le \|y\| \le 1 + \epsilon$. The following can be estimated by maximizing over the choice of x on the LHS,

$$|\langle f, [x]\rangle| \ = \ |\langle \tau(f), y\rangle| \ \le \ \|\tau(f)\|(1+\epsilon).$$

Minimizing over the choice of $\epsilon > 0$ on RHS above yields $\|f\| \le \|\tau(f)\|$. On the other hand, by taking

$$|\langle \tau(f), x\rangle| \ = \ |\langle f, [x]\rangle| \ \le \ \|f\|\|x\|$$

and maximizing over $x \in B$ with $\|x\| = 1$ yields $\|\tau(f)\| \le \|f\|$.

Thus, $\|\tau(f)\| = \|f\|$. By isometric property, one also gets injectivity. Therefore, τ is an isomorphism.

□

2.B Reflexivity

Note that a Banach space is reflexive if it is linearly isometric to its bidual under a canonical embedding. That is, a Banach space B is reflexive if the map $B \to B''$ given by

$$x \mapsto (x' \mapsto \langle x', x\rangle)$$

is surjective.

2.34 Proposition. *If a Banach space B is reflexive, then so is its bidual B''.*

Proof. Recall that $\iota : B \to B''$ is an isometric isomorphism and, by duality, so is $\iota' : B'' \to B'$, and so, $\iota'' : B'' \to B''''$. One still need to show that ι'' is the canonical embedding of B'' in B''''. For $x''' \in B'''$, one has by definition

$$\langle \iota''(x''), x''' \rangle \;=\; \langle x''', x'' \rangle \;=\; \langle x'', \iota'(x''') \rangle.$$

Next, ι is a canonical embedding, so

$$\begin{aligned}\langle x'', \iota'(x''') \rangle \;&=\; \langle \iota(\iota^{-1}(x'')), \iota'(x''') \rangle \\ &=\; \langle \iota'(x'''), \iota^{-1}(x'') \rangle.\end{aligned}$$

Using isomorphisms $\iota : B \to B''$ and $(\iota')^{-1} : B' \to B'''$ yields

$$\begin{aligned}\langle x'', \iota'(x''') \rangle \;&=\; \langle (\iota')^{-1}(\iota'(x''')), \iota(\iota^{-1}(x'')) \rangle \\ &=\; \langle x''', x'' \rangle.\end{aligned}$$

In conclusion, the map ι'' is indeed the canonical embedding of B'' in B''''. ☐

Next, one can relate the reflexivity of B to that of B' as in the following result.

2.35 Proposition. *Let B be a Banach space. Then B is reflexive if and only if its dual B' is reflexive.*

Proof. Assume B' is reflexive. Denote the canonical embeddings of B in B' by $\iota : B \to B''$ and $\iota_1 : B' \to B'''$. By assumption, ι_1 is invertible. Thus, for $x''' \in B'''$,

$$\begin{aligned}\langle x''', x'' \rangle \;&=\; \langle \iota_1(\iota_1^{-1}(x''')), x'' \rangle \\ &=\; \langle x'', \iota_1^{-1}(x''') \rangle.\end{aligned}$$

The surjectivity of ι must be ascertained. Let $x''' \in (\text{Range}(i))^{\perp}$. Since ι is an isometry, its range is closed and is equal to B''. Then by definition of $\perp$,

$$\begin{aligned}0 \;&=\; \langle x''', \iota(x'') \rangle \\ &=\; \langle \iota(x''), \iota_1^{-1}(x''') \rangle \\ &=\; \langle \iota_1^{-1}(x'''), x'' \rangle.\end{aligned}$$

The RHS vanishes for any argument in B'', hence $x''' = 0$. One concludes that

$$((\text{Range}(\iota))^{\perp})^{\perp} = B''$$

and so Range(ι) is dense in B''. Because ι is an isometry, it has closed range and thus it follows that B is reflexive.

Conversely, assume B is reflexive. Then by the preceding proposition, so is B''. Using the first part of this proof, one gets that B' is reflexive. ☐

Finally, it can be shown that reflexivity is inherited by closed subspaces of Banach space.

2.36 Proposition. *A closed subspace of a reflexive Banach space is reflexive.*

Proof. Let M be a closed subspace of reflexive Banach space B. The surjectivity of $\iota : M \to M''$ is what needs to be shown. Let $m'' \in M''$ and extend m'' to a functional $\sigma(m'')$ on B' by

$$\langle \sigma(m''), x' \rangle \;=\; \langle m'', x'|_M \rangle.$$

Note that $\sigma(m'') \in B''$, $x' \in B'$, $m'' \in M''$ and $x'|_M \in M'$. By reflexivity of B, $\sigma(m'') = \iota(x)$ for some $x \in B$. Doing so for each $m'' \in M''$ yields

$$\langle m'', x'|_M \rangle \;=\; \langle \iota(x), x' \rangle \;=\; \langle x', x \rangle.$$

Next, it needs to be shown that $x \in M$. If there is m'' for which $x \notin M$, then by Hahn-Banch Theorem ($M = \overline{M}$), there is $x' \in B'$ with $x'|_M = 0$ but $\langle x', x \rangle \neq 0$. Hence,

$$0 \;=\; \langle m'', x'|_M \rangle \;=\; \langle \iota(x), x' \rangle \;=\; \langle x', x \rangle \;\neq 0,$$

which is a contradiction. Therefore, $x \in M$. $\square$