Functional Analysis, Math 7321
Lecture Notes from February 9, 2017

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Recall that $\overline{B}_1^X$ and $\overline{B}_1^{X''}$ stand for the closed unit ball in X and X'', respectively, and that $i : X \to X''$ is the canonical embedding.

2.47 Theorem (Goldstine)**.** *Let X be a normed space. Then $\overline{B}_1^{X''} = \overline{i\left(\overline{B}_1^X\right)}^{w^*}$.*

2.48 Corollary. *Let X be a normed space. Then $i(X)$ is weak-* dense in X''.*

Proof. Observe that, by scaling,

$$X'' = \bigcup_{n=1}^{\infty} \overline{B}_n^{X''} = \bigcup_{n=1}^{\infty} \overline{i\left(\overline{B}_n^X\right)}^{w^*}.$$

Let $x \in X''$. Then there is an $n \in \mathbb{N}$ such that $x \in \overline{B}_n^{X''}$, which implies that $x \in \overline{i\left(\overline{B}_n^X\right)}^{w^*}$, which in turn implies that there are elements of $i\left(\overline{B}_n^X\right)$ arbitrarily close to x in the weak-* topology. Hence, $i(X)$ is weak-* dense in X''. □

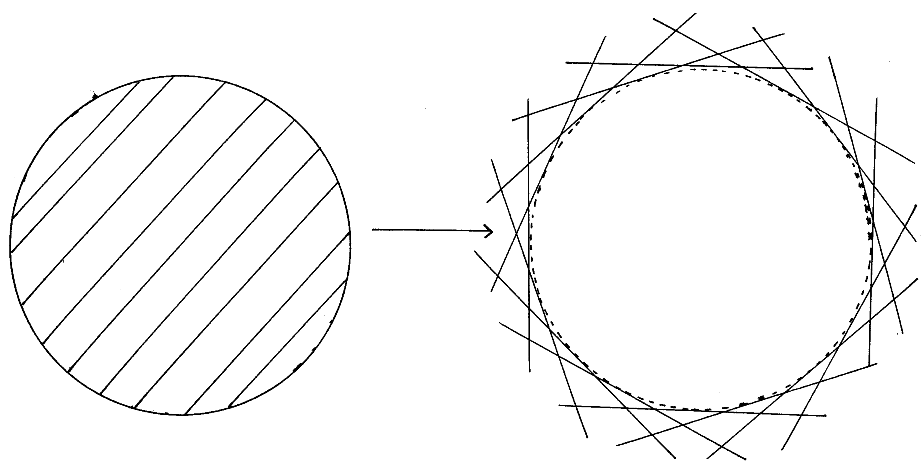
We now relate reflexivity to properties of $i|_{\overline{B}_1^X}$.

2.49 Lemma. $i\left(\overline{B}_1^X\right) = \overline{B}_1^{X''}$ *if and only if X is reflexive, that is, $i(X) = X''$.*

Proof. Suppose that $i\left(\overline{B}_1^X\right) = \overline{B}_1^{X''}$. Then, by scaling, $i(X) = \operatorname{span} i\left(\overline{B}_1^X\right) = \operatorname{span} \overline{B}_1^{X''} = X''$, where span is considered in terms of scalar multiplication.
Conversely, suppose that $i(X) = X''$ and let $f \in \overline{B}_1^{X''}$. Then there is an $x \in X$ such that $i(x) = f$. Since i is an isometry, it is the case that $\|x\| = \|f\| \leq 1$, which implies that $x \in \overline{B}_1^X$. Similarly, if $f \in i\left(\overline{B}_1^X\right)$, then there is an $x \in \overline{B}_1^X$ such that $i(x) = f$, which implies that $\|f\| = \|x\| \leq 1$, which in turn implies that $f \in \overline{B}_1^{X''}$. □

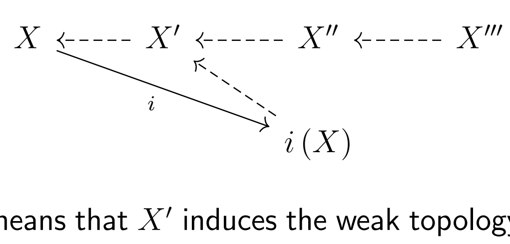
2.50 Theorem. *A normed space X is reflexive if and only if $\overline{B}_1^X$ is weakly compact.*

Proof. Suppose that i is reflexive. Then $i\left(\overline{B}_1^X\right) = \overline{B}_1^{X''}$. By the Banach-Alaoglu theorem, $\overline{B}_1^{X''}$ is weak-* compact. By the reflexivity of X, X'', and X', the initial topologies induced by $i_1(X')$ and X''' on X'' are identical. By reflexivity of X, the weak-* topology on $X'' = i(X)$ is identical to the weak topology induced by X' on X, provided that we identify X with X'' via the isomorphism i. Hence, $\overline{B}_1^X = i^{-1}\left(\overline{B}_1^{X''}\right)$ is weakly compact since it is isomorphic to $\overline{B}_1^{X''}$, which is weak-* compact.
Conversely, suppose that $\overline{B}_1^X$ is weakly compact. Note that $i|_{\overline{B}_1^X}$ is a homeomorphism onto $i\left(\overline{B}_1^X\right) \subseteq (X'', \tau_w^*)$, that is, X'' endowed with the weak-* topology, because given a net $(x_j)_{j\in J}$ in X, weak convergence of $(x_j)_{j\in J}$, that is, $x_j \xrightarrow{w} x \in \overline{B}_1^X$, is equivalent to $f(x_j) \to f(x)$ for each fixed $f \in X'$, which is equivalent to $i(x_j)(f) \to i(x)(f)$, or $i(x_j) \xrightarrow{w} i(x)$. Thus, $\overline{B}_1^X$ is weakly compact if and only if $i\left(\overline{B}_1^X\right)$ is weak-* compact in X'', in which case $i\left(\overline{B}_1^X\right)$ is closed since X'' is Hausdorff. By Goldstine, $i\left(\overline{B}_1^X\right) = \overline{i\left(\overline{B}_1^X\right)}^{w^*} = \overline{B}_1^{X''}$, and by the preceding lemma, $i(X) = X''$. Hence, i is reflexive. $\square$



This is an illustration of $\overline{B}_1^X$ being mapped to $\overline{B}_1^{X''}$ under i, where $\overline{B}_1^{X''}$ is being "carved out" by linear functionals on X''. Reflexivity asks whether these linear functionals "leave gaps" in terms of density.

We now summarize our insights on reflexivity.



In this diagram, $X' \dashrightarrow X$ means that X' induces the weak topology on X as an initial topology.

2.51 Theorem. *Let X be a Banach space. Then the following assertions are equivalent:*

(a) X *is reflexive.*

(b) On X'*, the weak topology induced by* X'' *is identical to the weak-** *topology induced by* $i(X)$*.*

(c) X' *is reflexive.*

Proof.
($(a) \implies (b)$) Suppose that X is reflexive. Then for any $f \in X''$, there is an $x \in X$ such that $i(x) = f$. Since i is a surjective isometry, it is the case that $\|f\| = \|x\|$, which implies that $i(X)$ and X'' induce the same topology on X'.
($(b) \implies (c)$) Suppose that the weak topology induced by X'' is identical to the weak-* topology induced by $i(X)$. Then by the Banach-Alaoglu theorem, $\overline{B}_1^{X'}$ is weak-* compact; by assumption, $\overline{B}_1^{X'}$ is weakly compact; and by the previous theorem, which characterizes reflexivity, X' is reflexive.
($(c) \implies (a)$) Suppose that X' is reflexive. Then it follows from the February 2 notes that X is reflexive. □

We now want to obtain sequential weak compactness of $\overline{B}_1^{X}$. So we prepare this with a result by Banach.

2.52 Theorem. *If* X *is a normed space and* X' *is separable, then* X *is separable.*

Proof. If $X = \{0\}$, then there is nothing to prove. Suppose that $X \neq \{0\}$ and let $S \subseteq X'$ be countable and dense. If $f \in S$, then there is an $x_f \in X$ such that $\|x_f\| = 1$ and $|f(x_f)| \geq \|f\|/2$. Let

$$S = \{f_j : j \in \mathbb{N}\} \qquad \text{and} \qquad L = \left\{ \sum_{i=1}^{m} c_i x_{f_i} : m \in \mathbb{N}, \operatorname{Re} c_i \in \mathbb{Q}, \operatorname{Im} c_i \in \mathbb{Q} \right\}.$$

Then L is countable.
We will show that L is dense in X, which is equivalent to showing that $L^\perp = \{0\}$ since $\left(L^\perp\right)^\perp = \{0\}^\perp = X$ and $\left(L^\perp\right)^\perp = \overline{L}$.
Suppose that $F \in X'$ satisfies $F|_L = 0$. Then by the density of S in X', we know that there is a sequence $\{g_n\}_{n=1}^\infty$ in S such that $g_n \to F$. More precisely, we know that $\|g_n - F\|_{X'} \to 0$. From the choice $x_f \in X$ for each $f \in S$, it follows that

$$\|g_n - F\| \geq |g_n(x_{g_n}) - F(x_{g_n})| = |g_n(x_{g_n})| \geq \frac{\|g_n\|}{2}.$$

As a result, $\|g_n - F\| \to 0$ implies that $\|g_n\| \to 0$, which in turn implies that $F = 0$. Therefore, L is dense in X. □

2.53 Remark. The reverse implication for separability is not true in general. For example, ℓ^1 is separable but ℓ^∞ is not (Recall that $(\ell^1)' = \ell^\infty$.).