Functional Analysis, Math 7321
Lecture Notes from February 14, 2017

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Warm up: Last week we covered local characterization of reflexivity. Separability of X' implies the separability of X. Now we want to show weak sequential compactness of the closed unit ball in the reflexive Banach space.

2.53 Theorem. *(Eberlein): Let X be reflexive Banach space, then the closed unit ball $\overline{B_1}^X$ is weakly sequentially compact.*

Proof. Let $\{x_n\}$ be a sequence in $\overline{B_1}^X$. Take $M = \overline{span}\{x_n : n \in \mathbb{N}\} \subset X$. Then we know that the closed subspace of reflexive Banach space is reflexive (by the theorem from last week). So M is reflexive (from Lecture note on February 2, 2017). Also, M is separable because the linear combinations of $\{x_n : n \in \mathbb{N}\}$ with rational coefficients are dense in M. Again by reflexivity of M, M'' is separable and by Banach theorem M' is also separable (from Lecture note on February 2, 2017). Now, by Helly selection principle (if X is separable Banach space, the $\overline{B_1}^{X'}$ in X' is w^*-sequentially compact(from Lecture note on February 7, 2017)) and weak-$*$ compactness, $\{i(x_n)\}_{n\in\mathbb{N}}$ in $\overline{B_1}^{M''}$ has weak-$*$ convergent subsequence say $\{i(x_{n_k})\}_{k\in\mathbb{N}}$.

Again by reflexivity, the weak and weak-$*$ topology in M'' coincide and weak-$*$ convergence of $\{i(x_{n_k})\}_{k\in\mathbb{N}}$ is equivalent to weak convergence of $(x_{n_k})_{k\in\mathbb{N}}$ in M, that is

$$x_{n_k} \stackrel{w}{\to} x, x \in M.$$

Next, given any $f \in X'$ we get

$$f(x_{n_k}) = f\big|_M(x_{n_k}).(since f\big|_M \in X' \quad then$$
$$f\big|_M(x_{n_k}) \to f\big|_M(x) = f(x)$$

Hece, $x_{nk} \stackrel{w}{\to} x$ in weak topology of X. Using that $\overline{B_1}^X = \overline{B_1}^{\overline{X}^w}$, by convexity, $x \in \overline{B_1}^X$.

□

This result has consequences for optimization problem

2.54 Corollary. *Let C be a non-empty closed convex subset of a reflexive Banach space X, then for each $x \in X$, there is $y \in C$ with*

$$\|x - y\| = \inf_{z\in C} \|x - y\|.$$

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a minimizing sequences, then

$$\|z_n - x\| \to \inf_{z\in C} \|x - z\|.$$

By the convergence of sequence of distances, there is $M > 0$ such that for each $n \in \mathbb{N}$

$$\|z_n\| \le \|z_n - x\| + \|x\| \le M.$$

So $(z_n)_{n\in\mathbb{N}}$ is in $\overline{B_M}^X$. We choose a weakly convergent subsequence according to Eberlein's theorem above, $z_{n_k} \overset{w}{\to} y$, so for each $f \in X', \|f\| \le 1$.

$$|f(x - y)| = \lim_{k\to\infty} |f(x - z_{n_k})|.$$

By using Hahn-Banach, we can choose f such that $|f(x - y)| = \|x - y\|$. This gives

$$\begin{aligned}
\|x - y\| &= \lim_{k\to\infty} |f(x - z_{n_k})| \\
&\le \lim_{k\to\infty} \|f\|.\|x - z_{n_k}\| \quad (since \|f\| = 1) \\
&= \inf_{z\in C} \|x - z\|
\end{aligned}$$

Finally, $y \in \overline{C}^w = \overline{C}$. So, $\inf_{z\in C} \|x - z\| \le \|x - y\|$ and the equality holds throughout. □

Another consequence of Eberlein's theorem:

2.55 Theorem. *Let X be reflexive Banach Space, and $T : X \to X$ be a continuous linear map such that $\sup_{n\in\mathbb{N}} \le \infty$, then*

$$\overline{T}x = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} T^j x$$

is defined for each $x \in X$, $\overline{T} \in B(X, X)$, $\overline{T}^2 = \overline{T}$ and $ran(\overline{T}) = \{x \in X : Tx = x\}$.

Proof. Let $A_n = \frac{1}{n}\sum_{j=1}^{n} T^j$ then by $\sup_{n\in\mathbb{N}} \|T^n\| = C < \infty$, we get $\|A_n\| \le \frac{1}{n}\sum_{j=1}^{n} \|T^j\| \le C$ (since $\|T^j\| \le C$. So $\{A_n\}_{n\in\mathbb{N}}$ are uniformly bounded.

We want to show that $\{A_n x\}_{n\in\mathbb{N}}$ converges for each fixed $x \in X$. We know $\|A_n x\| \le C\|x\|$ for each $n \in \mathbb{N}$. So $\{A_n x\}_{n\in\mathbb{N}}$ has weakly convergent subsequence $\{A_{n_k} x\}_{k\in\mathbb{N}}$.

Let y be the weak limit, we consider the sequence of differences

$$\begin{aligned}
x - A_n x &= x - \frac{1}{n}\sum_{j=1}^{n} T^j x \\
&= \frac{1}{n}\sum_{j=1}^{n} (I - T^j)x \\
&= \frac{1}{n}\sum_{j=1}^{n} (I - T)(I + T + T^2 + ... + T^{j-1})x \\
&= (I - T)\frac{1}{n}\sum_{j=1} (I + T + T^2 + ... + T^{j-1})x \in ran(I - T).
\end{aligned}$$

From above for any $n \in \mathbb{N}$, $x - A_n x \in ran(I-T) \subseteq \overline{ran}^w(I-T)$, and since y be the weak limit of $A_n x$ then

$$w - \lim_{n\to\infty}(x - A_n x) = (x-y) \in \overline{ran}^w(I-T).$$

Here, $ran(I-T)$ is convex set so by using Mazur theorem [1] we have,

$$\overline{ran}^w(I-T) = \overline{ran}(I-T)$$

Thus, we conclude,

$$w - \lim_{n\to\infty}(x - A_n x) = x - y \in \overline{ran}^w(I-T) = \overline{ran}(I-T).$$

Next, for any $x \in X$,

$$A_n(I-T)x = \frac{1}{n}(T - T^{n+1})x \to 0 \quad by \quad \|T - T^{n+1}\| \le C + C = 2C.$$

Again for all $f \in X'$,

$$\begin{aligned} \langle f, y\rangle &= \lim_{k\to\infty}\langle f, A_{n_k}x\rangle \\ &= \lim_{k\to\infty}\langle f, TA_{n_k}x\rangle \end{aligned}$$

By taking a adjoint of T,

$$\begin{aligned} \langle f, y\rangle &= \lim_{k\to\infty}\langle T'f, A_{n_k}x\rangle \\ &= \lim_{k\to\infty}\langle T'f, y\rangle \\ &= \langle f, Ty\rangle \end{aligned}$$

This is true for each f, so $Ty = y$.

Next,

$$\begin{aligned} T^j x &= T^j(y + x - y) \\ &= T^j(y) + T^j(x-y) \quad \text{(by linearity of } T^j) \\ &= y + T^j(x-y) \end{aligned}$$

and averaging over $j \in \{1,2,3,...,n\}$ gives $A_n x = y + A_n(x-y)$.

From $x - y \in \overline{ran}(I-T)$, for each $\epsilon > 0$ there is $w \in ran(I-T)$ and $\|x - y - w\| \le \epsilon/C$. Then

$$\|A_n(x-y)\| = \|A_n(x-y-w+w)\| \le \|A_n(x-y-w)\| + \|A_n w\| \quad \text{(by triangle inequality)}$$

Since, $\|A_n(x-y-w)\| < C.\frac{\epsilon}{C} = \epsilon$. Then for sufficiently large n,

$$\begin{aligned} \|A_n w\| &= \|A_n(I-T)h\| \\ &= \|\frac{1}{n}(T - T^{n+1})h\| \le \epsilon \end{aligned}$$

[1] Mazur Theorem: If C is convex set in a Normed space then its norm closure equals to its closure in weak topology

and thus,

$$\|A_n(x-y)\| < \epsilon + \epsilon = 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, so $\|A_n(x-y)\| = \|A_n x - y\| \to 0$ or equivalently $A_n x \to y$. Letting $\overline{T}x = \lim_{n\to\infty} A_n x = y$, we see from convergence that $\overline{T}$ is linear and from uniform boundedness $\overline{T} \in B(X, X)$ (from the note January 31, 2017).

Next,

$$\begin{aligned}(I-T)\overline{T}x &= (I-T)\lim_{n\to\infty} A_n x \\ &= \lim_{n\to\infty} \frac{1}{n}(T - T^{n+1})x = 0.\end{aligned}$$

This implies that $\overline{T} = T\overline{T}$.

Similarly, we can show that $\overline{T} = \overline{T}T$.

Finally, $T^j\overline{T}x = \overline{T}x$ for each x and $j \in N$ gives

$$\overline{T}^2 x = \lim_{n\to\infty} A_n \overline{T}x = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \overline{T}x = \overline{T}x.$$

Thus, $\overline{T}^2 = \overline{T}$. □