Functional Analysis II, Math 7321
Lecture Notes from February 23, 2017

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Last time, we saw that if T is invertible, then so is T'. Moreover, $(T')^{-1} = (T^{-1})'$. We continue the proof of the following proposition.

2.63 Proposition. *Given X a Banach space, Y is a normed space, $T \in B(X,Y)$ and T' is invertible. Then Y is a Banach space and T invertible.*

Proof. Last time, we had shown that $\inf_{\|x\|=1} \|Tx\| = \delta > 0$. By using the lower normed bound, we can show that $ran(T) = \{Tx : x \in X\}$ is closed. Let y_n be a Cauchy sequence in X. There is $x_n \in X$ such that $Tx_n = y_n$. If $x_n \neq x_m$,

$$\frac{\|Tx_n - Tx_m\|}{\|x_n - x_m\|} = \|T\big(\frac{x_n - x_m}{\|x_n - x_m\|}\big)\| \geq \delta.$$

Thus, for $x_n \neq x_m$,

$$\|x_n - x_m\| \leq \frac{1}{\delta}(\|Tx_n - Tx_m\|) = \frac{1}{\delta}(\|y_m - y_n\|).$$

The above inequality obviously holds for $x_n = x_m$. Since y_n is a Cauchy sequence, we obtain from the above inequality that x_n is also a Cauchy sequence. By the completeness of X, $x_n \to x$ for some $x \in X$. By the continuity of T, $Tx_n \to Tx$, i.e., $y_n \to Tx \in Y$ as $n \to \infty$. Thus, $ran(T)$ is complete; hence, it is closed. Let $F \in Y', F|_{T(X)} = 0$. Hence, for any $x \in X$, $F(Tx) = 0 = T'F(x)$. So, $T'F = 0$. By injectivity of T', $F = 0$. Thus, $(ran(T))^{\perp} = \{0\}$. Assume that $y \in Y \setminus ran(T) = \overline{ran(T)}$. By Hanh-Banach extension theorem, there exists a linear functional $G \in Y'$ such that $G(y) = 1$ and $G(z) = 0$ for all $z \in ran(T)$. Thus, $0 \neq G \in ran(T)^{\perp}$. This is a contradiction. Thus, $ran(T) = Y$. Thus Y is a Banach space since $ran(T)$ is complete. Using the open mapping theorem, T is open, hence, boundedly invertible. □

3 Operators on Banach Spaces

We study operators on a Banach space X over $\mathbb{C}$. We denote $B(X) = B(X,X)$, the set of all linear functions from X to itself. Since X is a Banach space, $B(X)$ is also a Banach space. Notice that for $T, S \in B(X)$, we have $\|T(S(x))\| \leq \|T\|\|S(x)\| \leq \|T\|\|S\|\|x\|$. Thus, $\|TS\| \leq \|T\|\|S\|$. This inequality is the principal property of a Banach space called "Banach algebra" and it is defined as follows.

3.1 Definition. A Banach algebra is a Banach space X with the multiplication such that for $x, y \in X$,

$$\|x \cdot y\| \leq \|x\|\|y\|.$$

We say that X is commutative if $x \cdot y = y \cdot x$ for every $x, y \in X$ and X is unital if it has an identity $1 \in X$ such that $1 \cdot x = x = x \cdot 1$ for every $x \in X$.

3.2 Examples.

- As we mentioned ealier, $B(X)$ is a Banach algebra with the composition as its multiplication.
- $C([0,1])$ the space of continuous functions on $[0,1]$ equipped with the sup norm and the pointwise multiplication is a Banach algebra [1, Chapter 4, Example 2]. In general, $C(X)$ when X is compact is a Banach algebra. The fact that $C(X)$ is a Banach space is well known. We will show that the pointwise multiplication is satisfied the inequality. Let $f, g \in C(X)$. Then,

$$\|fg\| = \sup_{x \in X} |f(x)g(x)| = \sup_{x \in X} |f(x)||g(x)| \leq \|f\| \sup_{x \in X} |g(x)| = \|f\|\|g\|.$$

- Let $L^1 = \{f : \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{\infty} |f(x)|dx < \infty\}$, the set of integrable functions. Then, L^1 is a Banach algebra [1, Chapter 4, Example 5] when we equip with pointwise addition, scalar multiplication and the norm

$$\|f\| = \int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

Define

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

This operation is called the convolution. The following shows the convolution satisfies the inequality. Consider

$$\begin{aligned}
\|f * g\| &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-y)g(y)dy \right| dx \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-y)||g(y)|dydx \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-y)||g(y)|dxdy \quad \text{(by Fubini's Theorem)} \\
&= \int_{-\infty}^{\infty} |g(y)| \int_{-\infty}^{\infty} |f(x-y)|dxdy \\
&= \|f\| \int_{-\infty}^{\infty} |g(y)|dy \\
&= \|f\|\|g\|.
\end{aligned}$$

Thus L^1 is a Banach algebra. However, it has no identity. Assume that f is the identity. Let χ_A be the characteristic function on a set A. Thus,

$$\chi_{[-1,1]}(x) = f * \chi_{[-1,1]}(x) = \int_{-\infty}^{\infty} f(x-y)\chi_{[-1,1]}(y)dy = \int_{-1}^{1} f(x-y)dy = \int_{x-1}^{x+1} f(y)dy.$$

Hence, $\chi_{[-1,1]}(1) = \int_0^2 f(y)dy = 1 = \int_{-2}^0 f(y)dy = \chi_{[-1,1]}(-1)$. Thus, $\int_{-2}^2 f(y)dy = 2$. However, $1 = \chi_{[-2,2]}(0) = f * \chi_{[-2,2]}(0) = \int_{-2}^2 f(y)dy$. This is a contradiction. Thus, L^1 has no identity.

3.3 Remark. If x_n converges to x and y_n converges to y, then $x_n y_n$ converges to xy. To see this, consider

$$\|x_n y_n - xy\| = \|x_n(y_n - y) - y(x - x_n)\| \leq \|x_n\|\|y_n - y\| + \|y\|\|x_n - x\|.$$

Since x_n is a convergent sequence, x_n is bounded. Moreover, $x_n \to x$ and $y_n \to y$ as $n \to \infty$ so $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$. Hence, $\|x_n y_n - xy\|$ converges to 0. This also proves that the multiplication operator in a Banach algebra is continuous in the product topology.

We will focus on studying the space $B(X)$ which is a remarkable Banach algebra. For $T \in B(X)$, we investigate the set $\{T - zI : z \in \mathbb{C}\}$.

3.4 Definition. Let X be a Banach space over $\mathbb{C}$ and $T \in B(X)$.

1. The resolvent set of T is

$$\rho(T) = \{z \in \mathbb{C} : T - zI \text{ is invertible}\}.$$

   If $z \in \rho(T)$, $R_z(T) = (zI - T)^{-1}$ is called the resolvent of T.

2. The spectrum of T is

$$\sigma(T) = \{z \in \mathbb{C} : T - zI \text{ is not invertible}\}.$$

If X is a X is a Banach space and $T \in B(X)$, we deduce from the proposition on the beginning of this note that T is invertible if and only if T' is invertible. The following result follows immediately from this fact.

3.5 Theorem. *For every $T \in B(X)$, $\sigma(T) = \sigma(T')$.*

Proof. Since X is a Banach space, $T - zI$ is invertible if and only if $(T - zI)' = T' - zI'$ is invertible. □

In addition, we notice that if $T \in B(X)$ which $\|T\| < 1$, then $I - T$ is invertible and $(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$. To prove this, let $S_n = \sum_{k=0}^{n} T^k$. Then, $\|S_m - S_n\| \leq \sum_{k=n+1}^{m} \|T\|^n$. Since $\|T\| < 1$, $\|S_n - S_m\| \to 0$ as $n, m \to \infty$. Thus, S_n is a Cauchy sequence in $B(X)$. Since $B(X)$ is complete, S_n converges to $S \in B(X)$ and $S(I - T) = \lim_{n\to\infty} \sum_{k=0}^{n} T^k(I - T) = \sum_{k=0}^{n}(I - T^{n+1}) = I + \lim_{n\to\infty} T^k$. Since $\|T\| < 1, \|T\|^n \to 0$ as $n \to \infty$. Thus, $T^k \to 0$. Thus, $S(I - T) = I$. Similarly, $(I - T)S = I$. We state this fact as the lemma.

3.6 Lemma. *Let $T \in B(X)$ where X is a Banach space. If $\|T\| < 1$, then $T - I$ is invertible and*

$$(T - I)^{-1} = -\sum_{k=0}^{\infty} T^k.$$

For $T \in B(\mathbb{R}^n)$, we know that $\sigma(T)$ is a non-empty finite set. The question arises, "In general, how $\sigma(T)$ behaves?" The next theorem answers the question.

3.7 Theorem (Gelfand)**.** *For each $T \in B(X)$, $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$.*

Proof. We divide the proof into steps.

Step 1: We show that $\sigma(T)$ is bounded. Let $r = \|T\|$. We claim $\sigma(T) \subseteq \overline{B_r} = \{z \in \mathbb{C} : |z| \leq r\}$. We write $T - zI = z(z^{-1}T - I)$. Since $\|z^{-1}T\| = |z|^{-1}\|T\| < 1$, by the Lemma above, $z^{-1}T - I$ is invertible. Hence, $T - zI$ is also invertible. Thus, $\mathbb{C} \setminus \overline{B_r} \subseteq \rho(T) = \mathbb{C} \setminus \sigma(T)$, i.e., $\sigma(T) \subseteq \overline{B_r}$.

Step 2: Next, we show $\rho(T)$ is open, so $\sigma(T)$ is closed. For this, we use that if $R \in B(X)$ is invertible, and S satisfies

$$\|S\| \leq \|R^{-1}\|^{-1}.$$

Then, $S - R$ is invertible. This is because if $V = R^{-1}S$, then $\|V\| \leq \|R^{-1}\|\|S\| < 1$. So, $\sum_{j=o}^{\infty} V^j$ converges to $(I - V)^{-1}$ by the Lemma. So $R - S = R(I - V)$ is invertible. We apply this to $R = T - zI$ and $S = (w - z)I$. Assuming $|w - z| < \|(T - zI)^{-1}\|^{-1}$. This shows $R - S = T - wI$ is invertible, hence, $B_r(z) \subseteq \rho(T)$ with $r = \|(T - zI)^{-1}\|^{-1} > 0$. Hence $\rho(T)$ is open. Thus, $\sigma(T)$ is closed and by step 1, it is also bounded. Thus, by Heine borel theorem, $\sigma(T)$ is compact.

Step 3: Finally, we show $\sigma(T) \neq \emptyset$. We assume that $\rho(T) = \mathbb{C}$. Then for $f \in B(X)'$, define $g : \mathbb{C} \to \mathbb{C}$,

$$g(z) = f((T - zI)^{-1}).$$

This map is well defined since $\rho(T) = \mathbb{C}$ and thus $(T - zI)^{-1}$ exists and bounded. We argue that g is holomorphic. Consider

$$\begin{aligned}
\frac{g(z+h) - g(z)}{h} &= \frac{f((T - zI - hI)^{-1}) - f((T - zI)^{-1})}{h} \\
&= \frac{f((T - zI - hI)^{-1} - (T - zI)^{-1})}{h} \\
&= \frac{f((T - zI - hI)^{-1}((T - zI) - (T - zI - hI))(T - zI)^{-1})}{h} \\
&= \frac{f((T - zI - hI)^{-1}(hI)(T - zI)^{-1})}{h} \\
&= f((T - zI - hI)^{-1}(T - zI)^{-1}).
\end{aligned}$$

As $h \to 0$, we obtain by continuity of f and the multiplication,

$$g'(z) = f((T - zI)^2).$$

Thus, g is differentiable and therefore, g is holomorphic. By Cauchy's integral formula, we have

$$g(z) = \frac{1}{2\pi i}\oint_{C_R} \frac{g(w)}{w-z} dw$$

where C_R is a circle centered at the origin with a radius $R > 0$ big enough so that it surrounds z. For $|w| > \|T\|$, $\|(T-wI)^{-1}\| = \|w^{-1}((1/w)T - I)\| = \|w^{-1}\sum_{k=1}^{\infty}\left(\frac{T}{w}\right)^k\| \le \frac{1}{|w|}\sum_{k=0}^{\infty}\left(\frac{\|T\|}{|w|}\right)^k = \frac{1}{|w|-\|T\|}$. Thus,

$$\begin{aligned}
|g(z)| &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{|g(Re^{i\theta})|}{|Re^{i\theta} - z|} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{|f((T - Re^{i\theta}I)^{-1})|}{|Re^{i\theta} - z|} d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \frac{\|f\|\|(T - Re^{i\theta}I)^{-1})\|}{|Re^{i\theta} - z|} d\theta \\
&\le \frac{\|f\|}{2\pi}\int_0^{2\pi} \frac{1}{|Re^{i\theta} - z|(|Re^{i\theta}| - \|T\|)} d\theta \\
&\le \frac{\|f\|}{2\pi}\int_0^{2\pi} \frac{1}{(R - |z|)(R - \|T\|)} d\theta \\
&= \frac{\|f\|R}{(R - |z|)(R - \|T\|)}.
\end{aligned}$$

As $R \to \infty$, we obtain $g(z) = 0$ for all $z \in \mathbb{C}$. Therefore, $f((T - zI)^{-1}) = 0$ for all $f \in B(X)$ which implies $(T - zI)^{-1} = 0$. This contradicts the invertibility of $T - zI$. We conclude that $\sigma(T) \neq \emptyset$. □

3.8 Remarks.

- We observe that by the Banach algebra property of $B(X)$, $\|T^n\| \le \|T\|^n$, and thus $\limsup_{n\to\infty}(\|T^n\|)^{1/n} \le \|T\| < \infty$. In step 1 in the proof, instead of $r = \|T\|$, we can improve $r = \limsup_{n\to\infty}(\|T^n\|)^{1/n}$. Let $z \in \mathbb{C}$ such that $|z| > r$, we can choose $\varepsilon > 0$ with $|z| > r + \varepsilon$ and for all sufficiently large n, $\|T^n\|^{\frac{1}{n}} < r + \varepsilon$. From the strict inequality, $|z| > r + \varepsilon$,

  $$\|z^{-n-1}T^n\| = \frac{1}{(r+\varepsilon)}\frac{\|T^n\|}{(r+\varepsilon)^n}\frac{(r+\varepsilon)^{n+1}}{|z|^{n+1}}.$$

  The norms decay exponentially and $-\sum_{n=0}^{\infty} z^{-n-1}T^n$ converges in norm to $S \in B(X)$. We see that $(T - zI)S = -(T - zI)\sum_{n=0}^{\infty} z^{-n-1}T^n = \lim_{N\to\infty} -(T - zI)\sum_{i=1}^{N} z^{-n-1}T^n = \lim_{N\to\infty}\sum_{i=1}^{N}(z^{-n}T^n - z^{-n-1}T^{n+1}) = z^0T^0 - \lim_{N\to\infty} z^{-N-1}T^{N+1} = I$. Same holds for $S(T - zI)$. So, S is the inverse of $T - zI$. We see $z \in \rho(T) = \mathbb{C} \setminus \sigma(T)$, and thus $\sigma(T) \subseteq \overline{B_r(0)} \subseteq \mathbb{C}$. We are going to see in the next lecture that there exists $z \in \sigma(T)$ which $|z| = \limsup_{n\to\infty}(\|T^n\|)^{1/n}$.

- We notice that the proof of the above theorem used the fact that $B(X)$ is a Banach space with identity. Therefore, the above theorem can be stated in more general setting as follows.

"Let X be a Banach algebra with identity. Then, for each $x \in X$, $\sigma(x)$ is a non-empty compact subset of $\mathbb{C}$."

References

[1] K. Chandrasekhara Rao *Functional Analysis, 2nd*, Alpha Science International, Oxford, 2006.