Functional Analysis, Math 7321
Lecture Notes from March 2, 2017

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Last time we continued our discussion of the spectrum of an operator T in $B(X)$ (where X is a Banach space) by showing that $\max\{|z| : z \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$. This justified calling $r = \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ the spectral radius of T. Before moving on to discussing projections and complemented subspaces, we examine one more example where we find the spectrum associated to a certain type of operator.

3.14 Example. **Multiplication Operators**

Given $f \in L^\infty([0,1])$, consider the operator $M_f : L^1([0,1]) \to L^1([0,1])$ defined by $M_f(g) = fg$. M_f is clearly linear, and we can show that:

$$\|M_f\| = \|f\|_\infty = \operatorname*{ess\,sup}_{x\in[0,1]} |f(x)| = \inf_{\alpha>0}\Big\{\alpha : m\big(\{x : |f(x)| > \alpha\}\big) = 0\Big\}.$$

To see this, first note that if $f = 0$ then we have nothing to show. So suppose $f \neq 0$, i.e. f is nonzero on a set of strictly positive measure. By Hölder's inequality,

$$\|M_f(g)\|_1 = \int_{[0,1]} |fg|\, dm \leq \int_{[0,1]} \|f\|_\infty |g|\, dm = \|f\|_\infty \|g\|_1.$$

Hence we have $\|M_f\| \leq \|f\|_\infty$. In particular, this tells us that M_f actually does map $L^1([0,1]) \to L^1([0,1])$. Conversely, define $A_n = \{x \in [0,1] : |f(x)| > \|f\|_\infty - \frac{1}{n}\}$. Let $g_n = \frac{\chi_{A_n}}{m(A_n)}$, and note $\|g_n\|_1 = 1$. By the definition of A_n we have for all $n \in \mathbb{N}$ and all $x \in A_n$:

$$\|f\|_\infty - \frac{1}{n} \leq |f(x)| \implies \int_{A_n} \left(\|f\|_\infty - \frac{1}{n}\right) dm \leq \int_{A_n} |f|\, dm.$$

By definition of g_n and the fact that $\|f\|_\infty - \frac{1}{n}$ is a constant, this gives:

$$\|f\|_\infty - \frac{1}{n} \leq \frac{1}{m(A_n)} \int_{A_n} |f|\, dm = \int_{[0,1]} |fg_n|\, dm = \|M_f(g_n)\|_1.$$

Taking the supremum over $n \in \mathbb{N}$ yields $\|f\|_\infty \leq \sup_{n\in\mathbb{N}} \|M_f(g_n)\|_1$. Note that the right hand side is bounded above by $\sup_{\|g\|=1} \|M_f(g)\|_1 = \|M_f\|$. Thus we have $\|f\|_\infty \leq \|M_f\|$, so equality holds.

Next, let $S = \{x \in \mathbb{C} : m\big(f^{-1}(B_r(z))\big) > 0 \text{ for all } r > 0\}$, called the *essential range* of f. We will show that the spectrum of M_f is exactly S.

If $z \in S$, let $D_n = f^{-1}(B_{\frac{1}{n}}(z))$ and $h_n = \frac{\chi_{D_n}}{m(D_n)}$. Then $\|h_n\|_1 = 1$, and:

$$\|(M_f - zI)(h_n)\|_1 = \int_{[0,1]} |fh_n - zh_n|\, dm = \frac{1}{m(D_n)} \int_{D_n} |f - z|\, dm \leq \frac{1}{n},$$

where the last inequality follows from the definition of D_n.

For the sake of contradiction, assume $M_f - zI$ had a bounded inverse L. Then for all $n \in \mathbb{N}$:

$$1 = \|h_n\|_1 = \|L(M_f - zI)(h_n)\|_1 \leq \|L\| \|(M_f - zI)(h_n)\|_1 \leq \frac{1}{n}\|L\|.$$

Thus $n \leq \|L\|$ for all $n \in \mathbb{N}$, contradicting that L was assumed to be bounded. So $M_f - zI$ is not invertible, hence $z \in \sigma(M_f)$. Since $z \in S$ was arbitrary, we have $S \subset \sigma(M_f)$.

To show the reverse inclusion $\sigma(M_f) \subset S$, suppose $z \notin S$. Then there exists some $r > 0$ with $m(f^{-1}(B_r(z))) = 0$. Define a function g on $[0,1]$ by setting $g(x) = \frac{1}{f(x)-z}$ if $x \in [0,1] \setminus f^{-1}(B_r(z))$, and $g(x) = 0$ elsewhere. Then g is measurable, and $\|g\|_\infty \leq \frac{1}{r}$. Hence on $[0,1] \setminus f^{-1}(B_r(z))$, $M_g(M_f - zI)(h) = h$, and since $m(f^{-1}(B_r(z))) = 0$, this means $M_g(M_f - zI)(h) = h$ almost-everywhere. Similarly we see $(M_f - zI)M_g(h) = h$ almost-everywhere.

Thus $M_f - zI$ is invertible, so $x \in \rho(M_f) = \mathbb{C} \setminus \sigma(M_f)$. Thus $\mathbb{C} \setminus S \subset \mathbb{C} \setminus \sigma(M_f)$. Taking complements yields $\sigma(M_f) \subset S$, so equality holds.

3.A Projections and Complemented Subspaces

In Hilbert spaces, there is a natural notion of projection onto closed subspaces due to the inner product structure. We now investigate the analog of projections in the more general setting of a Banach space. To begin with, we define what it means for a closed subspace to be complemented.

3.15 Definition. A closed subspace E of a Banach space X is called *complemented* if there is a closed subspace $F \subset X$ (called a *complementary subspace of* E) such that $E \cap F = \{0\}$ and $E + F = X$.

3.16 Remark. In a Hilbert space, every closed subspace E has a complement (namely, $E^\perp$). In a Banach space this is not automatic.

3.17 Remark. If E and F are complementary subspaces in a Banach space X, then if $x \in X$ can be written as $x = y + z$ where $y \in E$ and $z \in F$, then y and z are the unique such vectors.

Proof. Suppose $x = y_1 + z_1 = y_2 + z_2$ where $y_1, y_2 \in E$ and $z_1, z_2 \in F$. Then $y_1 - y_2 = z_2 - z_1$. Note that the left side of this equality is an element of E, and the right side is an element of F, hence $y_1 - y_2 = z_2 - z_1 \in E \cap F = \{0\}$. Hence $y_1 - y_2 = z_2 - z_1 = 0$, so $y_1 = y_2$ and $z_1 = z_2$. □

We next investigate some examples of complemented subspaces. As mentioned before, there *are* closed subspaces of Banach spaces that are not complemented. We focus first on types of subspaces that can be complemented, then mention a reference to an example of an uncomplemented subspace.

3.18 Examples.

(1) If $\dim(E) = n < \infty$, then E is complemented (note: finite-dimensional subspaces are automatically closed). To see this, let $\{x_1, \ldots, x_n\}$ be a basis for E. For each $i = 1, \ldots, n$, choose $f_i \in X'$ such that $f_i(x_j) = \delta_{i,j}$ (this is possible by the Hahn-Banach theorem). Let $F = \bigcap_{i=1}^n \ker(f_i)$, then F is a closed subspace and $E \cap F = \{0\}$ since $E = \operatorname{span}\{x_i\}_{i=1}^n$. It remains to show that $X = E + F$. Let $z \in X$ and define $x = \sum_{i=1}^n f_i(z)x_i$. Note that $x \in E$. Then for each $j = 1, \ldots, n$:

$$\begin{aligned} f_j(z - x) = f_j\left(z - \sum_{i=1}^n f_i(z)x_i\right) &= f_j(z) - \sum_{i=1}^n f_i(z) f_j(x_i) \\ &= f_j(z) - \sum_{i=1}^n f_i(z)\delta_{i,j} \\ &= f_j(z) - f_j(z) \\ &= 0. \end{aligned}$$

Thus $z - x \in \ker(f_j)$ for all $j = 1, \ldots, n$, hence $z - x \in F$. So we may write $z = x + (z - x)$, where $x \in E$ and $z - x \in F$. Since $z \in X$ was arbitrary, we have $X = E + F$. Thus F is a complementary subspace of E.

(2) If E is a closed subspace and $\operatorname{codim}(E) = \dim(X/E) = n < \infty$, then E is complemented. To see this, let $\{x_1 + E, \ldots, x_n + E\}$ be a basis for X/E, and define $F = \operatorname{span}\{x_1, \ldots, x_n\}$. Then F is closed because it is finite-dimensional.

If $z \in E \cap F$, then $z \in E$ so there exist constanst c_j such that $z = \sum_{j=1}^n c_j x_j$. Thus:

$$E = z + E = \sum_{j=1}^n (c_j x_j + E) = \sum_{j=1}^n c_j(x_j + E).$$

Since the coset E is the zero vector in X/E, and $\{x_j + E\}_{j=1}^n$ was a basis for X/E (hence linearly independent), this implies that each $c_j = 0$. Thus $z = 0$. So $E \cap F = \{0\}$.

Moreover, since $\{x_j + E\}_{j=1}^n$ is a basis for X/E there exist linear functionals $f_j \in (X/E)'$ such that $f_j(x_i + E) = \delta_{i,j}$. Note that if $z + E = \sum_{j=1}^n c_j(x_j + E) \in \bigcap_{j=1}^n \ker(f_j)$, then it follows that $c_j = 0$ for all j, hence $z \in E$. By linearity we have $E \subset \bigcap_{j=1}^n \ker(f_j)$, so equality holds.

Recall that $(X/E)'$ is isometrically isomorphic to $E^\perp \subset X'$ via the map $\tau : (X/E)' \to E^\perp$ given by $\tau(f)(x) = f(x+E)$. Define $g_j = \tau(f_j)$. Note that $g_j(x_i) = f_j(x_i + E) = \delta_{i,j}$, and for all $e \in E$ we have $g_j(e) = f_j(e+E) = f_j(E) = 0$.

Now, let $z \in X$, and let $x = \sum_{i=1}^n g_j(z)x_i \in F$. Then:

$$g_j(z-x) = g_j\left(z - \sum_{i=1}^n g_i(z)x_i\right) = g_j(z) - g_j(z) = 0.$$

So $z - x \in \bigcap_{j=1}^n \ker(g_j)$, hence $z - x + E \in \bigcap_{j=1}^n \ker(f_j) = E$. Thus $z = x + (z-x)$, with $x \in F$ and $z - x \in E$.

(3) Recall that c_0 (the space of all sequences which converge to 0) is a closed subspace of ℓ^∞. It was shown by Phillips that c_0 is not complemented in ℓ^∞ (see Phillips, *On Linear Transformations*).

We know if X is a Hilbert space then X is trivially a Banach space, and moreover all closed subspaces of X are complemented. What about the converse to this statement? We know that in general not all closed subspaces of a Banach space are complemented. Lindenstrauss and Tzafriri showed that the converse actually does hold, in other words: having complements for all closed subspaces is equivalent to being a Hilbert space.

3.19 Theorem. *(Lindenstrauss and Tzafriri,* "On the complemented subspaces problem"*)*
If X is a Banach space with complements for every closed subspace, then X is a Hilbert space (i.e., the norm of X is induced by an inner product).