Functional Analysis II, Math 7321
Lecture Notes from March 07, 2017

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Last Time: Spectrum of multiplication operators and complemented subspaces.

3.B More Examples of Complemented Subspaces

3. Every closed subspace E in a Hilbert space is complemented by $E^{\perp}$.

4. If
$$H^p = \overline{span}\{e^{2\pi inx}\}_{n=0}^{\infty} \in L^p([0,1]),\ 1 < p < \infty,$$
then H^p is complemented by
$$\overline{span}\{e^{2\pi inx}\}_{n<0}$$

It will be shown that projections do in fact provide an equivalent formulation of complemented subspaces.

3.20 Definition. An operator $P \in B(X)$ is a **projection** if $P^2 = P$.

3.21 Claim. *ker$P$ = ran$(I - P)$.*

Proof. If $Px = 0$, then $x = x - Px = (I - P)x$, and so if $x \in$ ker$P$, then $x \in$ ran(I-P). Conversely, if $y = (I - P)x$, then

$$Py = P(I - P)x = (P - P^2)x = Px - Px = 0.$$

So if $y \in$ ran$(I - P)$, then $y \in$ ker$P$. □

The following two theorems are the equivalent formulation of complemented subspaces provided by projections; the first theorem is the "easy" direction of such equivalence and part of the second is its converse.

3.22 Theorem. *If P is a projection, then ran$P$ is closed and complemented by ker$P$.*

Proof. Let $Q = I - P$. Then Q is a projection, ker$Q$ = ran$(I - Q)$ by the previous claim, and ran$(I - Q)$ = ran$P$. Morover, ran$P$ is complemented by ker$P$ = ran$(I - P)$ because if $x \in$ ran$P$ = ker(I-P) and $x \in$ ker, then

$$0 = (I - P)x = x.$$

Finally, for any $z \in X$, z is given by

$$z = Pz + (I - P)z$$

where $Pz \in$ ran$P$ and $(I - P)z \in$ ker$P$. □

3.23 Theorem. *A closed subspace E of X is complemented if and only if there is a projection $P \in B(x)$ such that $P^2 = P$ with E = ran$P$.*

Proof. If there is a projection, then by the theorem above, E = ran$P$ is complemented. Conversely, let F be a complementary subspace to E. If $z \in X$, with $x \in E$ and $y \in F$ being unique, one can write $z = x + y$. Let $Pz = x$, then by uniqueness, this is a well-defined linear map. Also, ran$P = E$ because if $x \in E$ and $0 \in F$, then

$$x = x + 0 \quad \Rightarrow \quad Px = x.$$

Moreover,

$$P^2 z = P(Pz) = Px = x = Pz, \quad Pz \in \text{ran}P.$$

Hence, $P^2 = P$.
To show $P \in B(X)$, consider the graph of $P \in X \oplus X$ with norm $||(z, x)|| = ||z|| + ||x||$, and let $(z_n, x_n) \to (z, x)$. Then $z_n = x_n + y_n$ where for each $n \in \mathbb{N}$, $x_n \in E$ and $y_n \in F$. So, $Pz_n = x_n \to x \in E$ by $\overline{E} = E$. Consequently, $y_n = z_n - x_n \to z - x \in F$ since F is closed. Thus, $z = x + (z - x)$ and $Pz = x$. Therefore, the limit is in the graph of P and hence P has a closed graph. Using the Closed Graph theorem, P is bounded. □

Complemented subspaces can also be used to study weak forms of invertiblity. Given Banach spaces X and Y, if $T \in B(X, Y)$, then T is said to be **left-invertible** if there is $S \in B(Y, X)$ such that $ST = I_X$.

3.24 Theorem. *Let X and Y be Banch spaces and $T \in B(X, Y)$. Then T is left-invertible if and only if T is injective and ran$T$ is closed and complemented.*

Proof. If T is injective and ran$T$ is closed and complemented, then taking P as the projection onto ran$T$,

$$T_0 = P \circ T : X \to \text{ran}T$$

is a projection onto a Banch space, so it is invertible by the open mapping theorem [W. Rudin, Theorem 2.11, (1)]. Hence, if $S = T_0^{-1}P$, then $ST = T_0^{-1}PT = I_X$ where $PT = T_0$. Therefore, T is left-invertible. On the other hand, if $S \in B(Y, X)$ is such that $ST = I_X$, then T is injective and

$$(TS)^2 = (TS)(TS) = T(ST)S = TS.$$

So, TS is a projection with ran$(TS) \subset$ ran$T$, but

$$\text{ran}T = \text{ran}(TST) \subset \text{ran}(TS)$$

and hence ran$T$ = ran(TS). Therefore, ran$T$ is the range of the projection that is closed and complemented 3.2.23. □

A second look at the result on ergodicity; recall that to show $A_n x = \frac{1}{n}\sum_{j=1}^{n} T^j x \to \overline{T}x$, one must assume power boundedness, $\sup_{n\in\mathbb{N}} ||T^n|| < \infty$.

3.25 Proposition. *If $T \in B(X)$ is power bounded, then $r(T) \leq 1$.*

Proof. From $||T^n|| \leq C$, one gets

$$\lim_{n\to\infty} ||T^n||^{\frac{1}{n}} \leq \lim_{n\to\infty} C^{\frac{1}{n}} = 1.$$

It was shown that $\overline{T}x = \lim_{n\to\infty} A_n x = y$ with $Ty = y$, equivalently, $y \in$ ker$(I-T)$. Conversely, if $y \in$ ker$(I-T)$, then $A_n y = y$ for each $n \in \mathbb{N}$, so $\overline{T}y = y$. Therefore, $\overline{T}(X) =$ ker$(I-T)$. □

One can also characterize the kernel of $\overline{T}$. From the statement on complementary projection, the ker$\overline{T}$ = ran$(I-\overline{T})$.

3.26 Corollary. *If $\overline{T}$ is as above, then the spaces E = ker$\overline{T}$ and F = ran$\overline{T}$ are complementary and F = ker$(I-T)$.*

References

[1] W. Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991.