Functional Analysis, Math 7320
Lecture Notes from March 09, 2017

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Warm-up: Finite rank operators are operators whose range is finite dimensional. In the settings of a Hilbert space $\mathcal{H}$, if T is a linear operator on $\mathcal{H}$ then T is finite rank (rank-n) if and only if there exist $v_1, v_2, ..., v_n$ and $e_1, e_2, ...e_n$ such that

$$Tx = \sum_{i=1}^{n} \langle x, v_i \rangle e_i$$

To see this, consider $dimranT = n$ and choose $e_1, e_2, ...e_n$ to be an Orthonormal Basis of $ranT$. Then

$$Tx = \sum_{i=1}^{n} \langle Tx, e_i \rangle e_i = \sum_{i=1}^{n} \langle x, T^* e_i \rangle e_i$$

The converse is also true. Thus, the rank one operators, whose action can be described as $Tx = \langle x, v \rangle e$ for some v and e, are the building blocks for the construction of finite rank operators.

Now, given a normed space X, what does a finite rank operator look like? For simplicity, let's study a rank-one operator, i.e. an operator $T \in B(X)$ such that $dimran(T) = 1$. Choosing any $y \in ran(T) \setminus \{0\}$, by definition of $ran(T)$, there exists some $x \in X$ such that $Tx = y$ and $x \notin kerT$. Now, pick a $g \in X'$ such that $g(y) = 1$. Then, for the operator $S : X \longrightarrow X$ defined as $Sx = T'(g)(x)y$, using the specific choices of y and g that we made, we have

$$\begin{aligned} Sx &= T'(g)(x)y \\ &= g(Tx)y \quad (by\ def\ of\ T') \\ &= 1 \cdot y \quad (by\ assumption) \\ &= Tx \quad (by\ assumption) \end{aligned}$$

thus, S has the same kernel as T. Moreover, $1 = dim(ranT) = dim(X) - dim(kerT)$, i.e. $kerT$ has co-dimension 1 and thus is complemented. So, by uniqueness of the analysis of any vector in X into components from $kerT$ and from a complement space of $kerT$, considering $(T'g)(x) = g(Tx)$ and $(T'h)(x) = h(Tx) = 0$ for all $h \in \{y\}^{\perp}$, we have characterized T', and this determined T uniquely.

To treat the $rank - r$ case, consider $y \in ran(T) \setminus \{0\}$ and g as above, let

$$S = T - (g \circ T)y$$

and show, by induction, that $dimran(S) \leq r - 1$.

Compact Operators

Finite rank operators in $B(X)$ are understood via Jordan form. Can we generalize the Jordan block form in infinite dimensions? To address this question we introduce the concept of a *compact operator*, which will serve as a natural generalization of a finite-rank operator in infinite dimensions.

3.27 Remark. We recall that

1. If V is a normed space and the closed unit ball $\overline{B_1}(0)$ is a compact neighborhood of 0, then $dim V < \infty$.

2. In a metric space, a set A is compact if and only if it is sequentially compact, while this in turn is equivalent to A being complete and totally bounded (i.e. for any $\epsilon > 0$ there exists a finite ϵ-net in A).

3. If in a metric space a set A is totally bounded, then so is $\overline{A}$.

4. If M is a compact metric space, $A \subset C(M)$ is sequentially compact if and only if A is closed, bounded and equicontinuous, i.e. for $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in M$ with $d(x, y) < \delta$ and $f \in A$ we have $|f(x) - f(y)| < \epsilon$.

3.28 Definition. Let X, Y be Banach spaces. A (bounded) linear map $T : X \longrightarrow Y$ is compact if $\overline{T(B_1^X(0))}$ is compact in Y[1].

3.29 Remarks. 1. Recall that, a subset of a complete metric space is totally bounded if and only if it is relatively compact (i.e. its closure is compact). Thus, the linear map $T : X \longrightarrow Y$, between the Banach spaces X and Y, is compact if $T(B_1^X(0))$ is totally bounded.

2. By equivalence of sequential and compactness, if $\{x_n\}_{n=1}^{\infty}$ is bounded in X, then $\{Tx_n\}$ has a convergent subsequence.

3. If $\overline{T(B_1^X(0))}$ is compact and $ran(T)$ is infinite dimensional, then $ran(T)$ cannot contain $B_\epsilon^Y(0)$ for all $\epsilon > 0$, otherwise $\overline{T(B_1^X(0))}$ would be a compact neighborhood of 0.

The next result allows us to enlarge the set of compact operators from what we know it to be.

3.30 Theorem. *Let X, Y, Z be Banach.*

(a) If $T \in B(X, Y)$ is finite rank, then it is compact.
If $T \in B(X, Y)$ is compact, then $ran(T)$ cannot contain infinite-dimensional closed subspaces.

(b) If T_1, T_2 are compact and $c \in \mathbb{K}$ then $T_1 + cT_2$ is compact.

(c) If $T \in B(X, Y)$ is compact and $S \in B(Y, Z)$ then ST is compact.

(d) If $S \in B(Y, Z)$ is compact and $T \in B(X, Y)$ then ST is compact.

[1]alternatively, one might say that $T(B_1^X(0))$ is relatively compact in Y

(e) If $T \in B(X,Y)$ is compact and invertible then $dimX = dimY < \infty$

(f) If V is closed subspace of X and $T \in B(X,Y)$ is compact, then $T\big|_V \in B(V,Y)$ is compact.

(g) If $T \in B(X,Y)$ is compact, then $ran(T)$ is separable.

(h) If $\{T_n\}_{n=1}^{\infty}$, where each T_n is compact and $T_n \longrightarrow T$ in operator norm, then T is compact.

The converse of (h) is reffered to as the *Compact Approximation Property (CAP)*, and does not hold for Banach spaces in general. It might also be worth noting here that, the limit of finite-rank operators between Banach spaces is again always a compact operator. But once more, the converse, also known as the *Aproximation Property (AP)*, though true for Hilbert spaces, does not hold for general Banach spaces[2]. For example ℓ^p for $p \neq 2$ as well as c_0 contain closed subspaces that do not satisfy this property. Furthermore, Willis in 1992 presented the construction[3] of a Banach space that does not have the AP even though it has the CAP, proving that CAP does not imply the AP.

Now, before we proceed with the proof of the Theorem's properties, let's first use them to construct some examples of compact operators.

3.31 Examples. (1) Let $X = Y = \ell^p$ and $a \in c_0$. Define M_a as

$$M_a(x_1, x_2, ...) = (a_1 x_1, a_2 x_2, ...)$$

so that

$$||M_a|| \leq \sup_{n \in \mathbb{N}} |a_n| < \infty$$

Also, for $n \in \mathbb{N}$, let T_n be defined as

$$T_n(x_1, x_2, ..., x_n, x_{n+1}, ...) = (a_1 x_1, a_2 x_2, ..., a_n x_n, 0, 0, ...)$$

Then T_n is a finite rank operator, hence compact by (a). Furthermore

$$||M_a - Tn|| \leq \sup_{m > n} |a_m| \stackrel{n \to \infty}{\longrightarrow} 0$$

thus, by (h), M_a ia also compact.

(2) Compact operators arose in the theory of integral equations, since integral operators provide concrete examples of such operators. Approximation by finite-rank operators is an essential tool in the numerical solution of such equations. The concept of a *Fredholm operator* is introduced in these settings. Let $X = Y = C([0,1])$, equipped with the $max - norm$, and let $G \in C([0,1]^2)$. Let the Fredholm operator associated with G be

$$(Tf)(x) = \int_0^1 G(x,y) f(y) dy$$

[2]Banach stated this problem in his book (1932) and it remained an un-solved problem for quite a few years, until the Swedish mathematician Per H. Enflo published the first counterexample in 1973.

[3]the construction is based on results of Grothendieck who did significant work in this area during his, hopeless, effort to prove that the AP holds for Banach spaces in general.

Observe how this operator can be perceived as a generalization of linear mappings (matrices) in infinite dimensions[4].
Now, if $G(x,y) = F(x)H(y)$ for $F, H \in C([0,1])$, then $dimran(T) \leq 1$, so the corresponding T is rank-one and thus compact.
If $G(x,y) = \sum_{j=1}^{r} F_j(x)H_j(y)$, then T is rank-r, hence compact.
By Stone-Weierstrass, any $G \in C([0,1]^2)$ can be approximated uniformly by (polynomial) functions of this type, thus T is approximated in norm by finite-rank Fredholm operators. Hence, T is compact.

(3) Let $X = Y = L^2([0,1])$, $G \in L^2([0,1]^2)$ and T defined as above (in (2)). Then, for any $f \in L^2([0,1]^2)$ we have

$$\begin{aligned}
||Tf||_2^2 &= \int_{[0,1]} |Tf(x)|^2 dx \\
&= \int_{[0,1]} |\int_{[0,1]} G(x,y)f(y)dy|^2 dx \\
&\overset{C.S}{\leq} \int_{[0,1]} \left(\int_{[0,1]} |G(x,y)|^2 dy \right) \left(\int_{[0,1]} |f(y)|^2 dy \right) dx \\
&= ||G||_2^2 ||f||_2^2
\end{aligned}$$

[4] if we consider G as being an $n \times n$ matrix, then the operation of G acting on some n-dimensional vector x can be written as

$$Gx(i) = \sum_{j=1}^{n} G(i,j)x(j)$$

which is an obvious discrete analogue of the Fredholm operator we defined