Functional Analysis II, Math 7321
Lecture Notes from March 21, 2017

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We begin with more examples on compact operators.

(3) Let $X = Y = L^2([0,1])$, and $G \in L^2([0,1]^2)$. We consider the Hilbert-Schmidt integral operator

$$Tf(x) = \int_0^1 G(x,y)f(y)dy.$$

We had shown $\|Tf\|^2 \leq \|G\|_2^2 \|f\|_2^2$, using the Cauchy-Schwarz inequality. In the same fashion, if $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of such operators with integral kernels $\{G_n\}_{n\in\mathbb{N}}$, then

$$\|T - T_n\| \leq \|G - G_n\|_2.$$

Since continuous functions are dense in $L^2([0,1]^2)$, each T corresponding to an integral kernel $G \in L^2([0,1]^2)$ is the limit of a sequence of compact operators. To prove this, using the Stone-Weierstrass theorem, the kernel G can be uniformly approximated in $[0,1]^2$ by linear combinations of maps of form $(x,y) \mapsto u(x)v(y)$, where $u, v : [0,1] \to \mathbb{R}$ are continuous functions. Then, for such functions, the Hilbert-Schmidt integral operator has finite rank. Thus, by what the Cauchy-Schwarz inequality implied above, if $\{T_n\}_{n=1}^{\mathbb{N}}$ is a sequence of Hilbert-Schmidt operators with kernels that approximate uniformly the kernel of T, we obtain a sequence of compact operators $\{T_n\}$ satisfying $T_n \to T$ in the operator norm. Thus, T is compact.

(4) Equipped with the norm $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$, let X be the Banach space $X = C^1([0,1])$. Let $Y = C([0,1])$ and let T be the inclusion map $T : X \to Y$ given by $Tf = f$. Then T is compact because if $\{f_n\}_{n\in\mathbb{N}} \subset C^1([0,1])$ satisfies $\|f_n\|_{C^1} \leq 1$, then it must be that $\|f_n\|_\infty \leq 1$ and $\underline{\|f_n'\|_\infty \leq 1}$, which means $\{f_n\}_{n=1}^\infty$ is bounded and equicontinuous in $C([0,1])$ and hence $\overline{T(B_1(0))}$ is compact.

3.32 Theorem. *Let X, Y and Z be Banach spaces. Then*

(a) If $T \in B(X,Y)$ is finite rank, then T is compact. If $T \in B(X,Y)$ is compact, then $ran(T)$ cannot contain infinite-dimensional closed subspaces.

(b) If T_1, T_2 are compact anc $c \in \mathbb{K}$, then $T_1 + cT_2$ is compact.

(c) If $T \in B(X,Y)$ is compact and $S \in B(Y,Z)$, then ST is compact.

(d) If $S \in B(X,Y)$ is compact and $T \in B(X,Y)$, then TS is compact.

(e) If $T \in B(X,Y)$ is compact and invertible, then $dim(X) = dim(Y) < \infty$.

(f) If V is a closed subspace of X and $T \in B(X,Y)$ is compact, then $T|_V \in B(V,Y)$ is compact.

(g) If $T \in B(X,Y)$ is compact, then $ran(T)$ is separable.

(h) If $\{T_n\}_{n=1}^{\infty}$, where each T_n is compact and $T_n \to T$ in operator norm, then T is compact.

Proof. (a) Since the range of T is finite dimensional, the closure of the unit ball in the range of T,

$$\overline{B_1}^{ran(T)} = \overline{B_1^{ran(T)}},$$

is compact. Also, since $\|T\| \leq C < \infty$, we have $\overline{T(B_1^X)} \subset C\overline{B_1^{ran(T)}}$, so the closure of $T(B_1^X)$ is compact, as it is a closed subset of a compact set. Thus, T is compact.

Moreover, by continuity of T, if Z is a closed subspace in $ran(T)$, then $W = T^{-1}(Z)$ is also closed. Now consider $T|_W : W \to Z$. By the open mapping theorem, there is $\epsilon > 0$ with $B_\epsilon^Z \subset T|_W(B_1^W)$, while by compactness of T,

$$\overline{T(B_1^W)} \subset \overline{T(B_1^X)}.$$

This means $\overline{T(B_1^W)}^Z$ is a compact neighborhood of 0 in Z. Hence, Z must be finite dimensional.
(b) If T_1 and T_2 are compact, then $\overline{T_1(B_1^X)}$ and $c\overline{T_2(B_1^X)}$ are both compact. Moreover,

$$\overline{(T_1 + cT_2)(B_1^X)} \subset \overline{T_1(B_1^X)} + c\overline{T_2(B_1^X)}$$

and since $g : X \times X \to X$, $g(x,y) = x + cy$ is continuous and maps the compact sets to compact sets, the RHS is compact. Thus, $\overline{(T_1 + cT_2)(B_1^X)}$ is compact, as it is closed subset of a compact set.
(c) If T is compact and S is bounded, then $ST(B_1^X) \subset S(\overline{T(B_1^X)})$. Then $\overline{T(B_1^X)}$ is compact and S is continuous, so $S(\overline{T(B_1^X)})$ is compact. Moreover, $\overline{ST(B_1^X)} \subset S(\overline{T(B_1^X)})$, and so $\overline{ST(B_1^X)}$ is compact.
(d) If T is compact and S is bounded, then

$$TS(B_1^X) \subset T(\|S\|B_1^Y) \subset \|S\|T(B_1^Y) \subset \|S\|\overline{T(B_1^Y)},$$

and so since $\|S\|\overline{T(B_1^Y)}$ is compact and $\overline{TS(B_1^X)} \subset \|S\|\overline{T(B_1^Y)}$, we have $\overline{TS(B_1^X)}$ is compact.
(e) Be (c) and (d), if $T^{-1}T = I_X$, and $TT^{-1} = I_Y$, and if both are compact, $\overline{B_1^X}$ and $\overline{B_1^Y}$ are compact, thus both X and Y are finite dimensional.
(f) We have $B_1^V = B_1^X \cap V$ and $T|_V(B_1^X \cap V) \subset \overline{T(B_1^X)}$. The RHS is compact, so $\overline{T|_V(B_1^X \cap V)}$ is a closed subset of a compact set, and therefore compact.
(g) Let $\epsilon_n = 1/n$, for $n \in \mathbb{N}$. We have $T(B_1^X)$ is totally bounded. Then, for each such ϵ_n there is a finite set of points $A_n = \{x_1^n, \ldots, x_{M_n}^n\} \subset T(B_1^X)$ such that

$$T(B_1^X) = \bigcup_{k=1}^{M_n} B_{1/n}(x_k).$$

We consider $A = \cup_{n\in\mathbb{N}} A_n$. Then A is countable, while for any arbitrary point $x \in T(B_1^X)$ and for any $n \in \mathbb{N}$, there is some $y_n \in A_n$ so that $x \in B_{1/n}(y_n)$. This argument generates a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \to x$. So x is in the closure of A and so A is dense in $T(B_1^X)$, which means the latter is separable. Moreover, $ran(T) = \cup_{n=1}^{\infty} nT(B_1^X)$, so $ran(T)$ is also separable, as it is the union of separable spaces.
(h) If $T(B_1^X)$ is totally bounded, then $\overline{T(B_1^X)}$ is totally bounded. For this, given $\epsilon > 0$, we can find $\{x_1, \dots, x_M\} \subset T(B_1^X)$ such that $T(B_1^X) \subset \cup_{k=1}^M B_{\epsilon/2}(x_k)$. Then, for an arbitrary $z \in \overline{T(B_1^X)}$, there exists $x \in T(B_1^X)$ such that $\|z - x\| < \epsilon/2$ and there is some k such that $\|x - x_k\| < \epsilon/2$, which means $\|z - x_k\| < \epsilon$. Hence, $\overline{T(B_1^X)} \subset \cup_{k=1}^M B_\epsilon(x_k)$.

Next, given $\epsilon > 0$, let N be such that for all $n \geq N$, $\|T_n - T\| < \epsilon/3$, with each T_n compact. $\overline{T_n(B_1^X)}$ is totally bounded, by compactness. Thus, for n fixed, there is a finite set $\{x_1, \dots, x_n\} \subset B_1^X$, with $\{T_n x_1, \dots, T_n x_m\}$ being an $\epsilon/3$-net for $T_n(B_1^X)$. Thus, for any $y \in B_1^X$, there is j with $\|T_n y - T_n x_j\| < \epsilon/3$. Consequently,

$$\|Ty - Tx_j\| \leq \|Ty - T_n y\| + \|T_n y - T_n x_j\| + \|T_n x_j - Tx_j\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

We therefore see $\{Tx_1, \dots, Tx_m\}$ is an ϵ-net for $T(B_1^X)$ and so $T(B_1^X)$ is totally bounded. $\square$

3.33 Remark. We have shown limits of finite rank operators are compact, but is the converse true? In Hilbert spaces, the answer is yes. However, this is not true in general and we prepare this with another characterization of compactness.

3.34 Theorem. *If $T \in B(X,Y)$ is compact, then it is completely continuous, i.e, for each weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \rightharpoonup x$, $Tx_n \to Tx$ in norm. For reflexive X, the converse is true as well.*

Proof. Assume $x_n \rightharpoonup x$, but $Tx_n \not\to Tx$. There is $\epsilon > 0$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\|Tx_{n_k} - Tx\| \geq \epsilon,$$

for all $k \in \mathbb{N}$. By uniform boundedness, weak convergence of $\{x_{n_k}\}_{k\in\mathbb{N}}$ implies boundedness (see Theorem 0.0.4 below). Hence $\{x_{n_k}\}_{k\in\mathbb{N}} \subset B_r^X$, for some $r > 0$ and there is a subsequence $\{x_{n_{k_j}}\}_{j\in\mathbb{N}}$ such that $Tx_{n_{k_j}} \to z$ in norm. But $x_n \rightharpoonup x$, so for $f \in Y'$,

$$f(Tx_{n_{k_j}} - Tx) = T'(f)(x_{n_{k_j}} - x) \to 0.$$

We conclude $Tx_{n_{k_j}} \rightharpoonup Tx = z$, since Y' separates points in Y, a contradiction.

For the converse, assume X is reflexive. Then for every bounded sequence $\{x_n\}_{n\in\mathbb{N}}$, there is $\{x_{n_k}\}_{k\in\mathbb{N}}$, converging weakly to some w. Then, by complete continuity, $Tx_{n_k} \to Tw$ and hence we have found a convergent subsequence, and thus T is compact. $\square$

3.35 Theorem. *Let X be a Banach space. Every weakly convergent sequence in X is bounded.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in X. In addition, define $T_n \in X''$ by $T_n(g) = g(x_n)$ for all $g \in X'$. Then, for a fixed $g \in X'$ and for any $n \in \mathbb{N}$, since the sequence $\{g(x_n)\}_{n\in\mathbb{N}}$ is convergent, the set $\{T_n(g)\}$ is bounded. Thus, by uniform boundedness,

$$\sup_{n\in\mathbb{N}} \|x_n\| = \sup_{n\in\mathbb{N}} \|T_n\| < \infty,$$

which means $\{x_n\}_{n\in\mathbb{N}}$ is bounded. $\square$

References

[1] W. Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991.