Functional Analysis II, Math 7321
Lecture Notes from March 28, 2017

taken by Robert P Mendez

3.40 Remark. Recall that we defined the compactness of $T \in B(X,Y)$ in terms of $\overline{T(B_1(0))}$ [Notes, March 9]. However, we could have equivalently defined T to be compact if $\overline{T(\overline{B}_1(0))}$ is compact in Y; we showed that compactness of the closure of the image of the unit ball is equivalent to the condition that any bounded sequence has a strongly convergent subsequence, using a bound of 1, but we may set the bound to $1+\epsilon$ to demonstrate the equivalence of the stronger formulation. Since $\overline{T(\overline{B}_1(0))} \supset \overline{T(B_1(0))}$, it may be more difficult to prove compactness in terms of the larger set.

Warm-up

We note that we may have incidentally used the following result without having shown it.

3.41 Claim. *If $T \in B(X,Y))$ is compact, them T' is compact.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y' with $\|y_n\| \leq 1$ for all $n \in \mathbb{N}$. Considering $\overline{B}_1(0) \subset X$, T compact implies $W := \overline{T(\overline{B}_1(0))}$ is compact in Y. Thus, we may apply Ascoli's Theorem [A5, (1)] once establishing that $(y_n)_{n\in\mathbb{N}}$ is an equicontinuous family on W

Given $x, z \in W$, we have $|y_n(x) - y_n(z)| = |y_n(x-z)|$ by linearity, which can only increase by taking the norm of y_n, so $|y_n(x-z)| \leq \|y_n\|\|x-z\| \leq \|x-z\|$. Thus, $(y_n)_{n\in\mathbb{N}}$ is a uniformly bounded equicontinuous family in $C(\overline{T(\overline{B}_1(0))})$. The corollary to Ascolli's Theorem provides a uniformly convergent subsequence $(y_{n_k})_{k\in\mathbb{N}}$ in $C(\overline{T(\overline{B}_1(0))})$. Since T is continuous and $T'y_{n_k} = y_{n_k} \circ T$, the sequence $(T'y_{n_k})_{k\in\mathbb{N}}$ is uniformly convergent in $C(\overline{B}_1(0))$. By the definition of the norm in X', this means $(T'y_{n_k})_{k\in\mathbb{N}}$ is convergent in norm. Thus, we have found a norm convergent subsequence of $(T'y_{n_k})_{k\in\mathbb{N}}$, and T' is compact. □

3.42 Theorem (Riesz-Fredholm). *Let $T \in B(X)$ be compact, $c \neq 0$, $N_j = \ker(T-cI)^j$ and $M_j = \operatorname{ran}(T-cI)^j$; then the following hold:*

(a) $N_1 \subset N_2 \subset \cdots$, and there exists $k_N \in \mathbb{N}$ such that $N_{k_N-1} \subsetneq N_{k_N} = N_{k_N+j}$ for all $j \in \mathbb{N}$, and all N_j are invariant under T and of finite dimension;

(b) $M_1 \supset M_2 \supset \cdots$, and there exists $k_M \in \mathbb{N}$ such that $M_{k_M-1} \supsetneq M_{k_M} = M_{k_M+j}$ for all $j \in \mathbb{N}$, and all M_j are invariant under T, closed and of finite codimension;

(c) $k_N = k_M =: k$ *and* $M_,$ *and* N_k *are complementary (closed) subspaces. In addition,* $(T-cI)|_{M_k}$ *is invertible in* $B(M_k)$*, and* $(T-cI)|_{N_k}$ *is nilpotent in* $B(N_k)$ *with index* k*, meaning* $(T-cI)^{k-1}\big|_{N_k} \neq 0$ *and* $(T-cI)^k\big|_{N_k} = 0$*.*

(d) $\dim\ker(T-cI) = \operatorname{codim}\operatorname{ran}(T-cI)$

Proof. (a) The inclusion is clear from composition, and we now show that each N_j is invariant under T: fixing j and taking $x \in N_j$, we note that T commutes with each of the terms of $(T-cI)^j$. Thus $(T-cI)^j Tx = T(T-cI)^j x$, which equals 0 since $x \in N_j$, and $Tx \in N_j$.

To show the existence of k_N as in the statement of the theorem, we demonstrate that it is enough to show that $N_k = N_{k+1}$ for some $k \in \mathbb{N}$: tSuppose such a k exists, and take k_N to be the least such index. Then for $x \in N_{k_N+2}$, we have that $(T-cI)x \in N_{k_N+1} = N_{k_N}$, which implies $x \in N_{k_N+1}$ and $N_{k_N+1} = N_{k_N+2}$. An induction argument yields that $N_{k_N} = N_{k_N+j}$ for all $j \in \mathbb{N}$. Thus, k_N exists if $N_k = N_{k+1}$ for some k.

In pursuit of a contradiction, suppose that for each $j \in \mathbb{N}$, $N_j \subsetneq N_{j+1}$. We note that each N_j is closed, as the kernel of a bounded operator, and we produce a sequence $(x_j)_{j\in\mathbb{N}}$ by choosing $x_j \in N_j \backslash N_{j-1}$ for each j such that $x_j + N_{j-1} \in N_j/N_{j-1}$ and $\|x_j + N_{j-1}\| = 1$.

For $j > 1$, we choose $y_j \in N_{j-1}$ such that $\|x_j + y_j\| \leq 2$; the existence of such points follows from the definition of the quotient norm an infimum of distances. Setting $x'_j := x_j + y_j$, we have that $x'_j + N_j = x_j + N_j$, $\|x'_j\| \leq 2$, and $\|x'_j + N_{j-1}\| = 1$ for each $j \in \mathbb{N}\backslash\{1\}$.

Consider, for $i < j \in \mathbb{N}$, the differences $Tx'_j - Tx'_i$, which we may artificially write as

$$\begin{aligned} Tx'_j - Tx'_i &= cx'_j + (Tx'_j - cx'_j) - Tx'_i \\ &\in\ cx'_j + \quad N_{j-1} \quad + N_i, \qquad \left(\substack{\text{since } Tx'_j - cx'_j = (T-cI)x'_j \\ \text{and } (T-cI)^{j-1}(T-cI)x'_j = 0}\right) \\ &=\ cx'_j + N_{j-1} \\ &= c(x'_j + N_{j-1}) \\ &= c(x_j + N_{j-1}). \end{aligned}$$

Since $\|x_j + N_{j-1}\| = 1$, it follows that $\|Tx'_j - Tx'_i\| \geq \|c(x_j + N_{j-1})\| = |c| > 0$ for all $i < j \in \mathbb{N}$. Thus $(Tx'_j)_{j\in\mathbb{N}}$ has no convergent subsequence, which is absurd in the light of T's compactness. We conclude that our supposition was false, and the existence of k_N is shown.

Finally, since T is compact, so is

$$\begin{aligned} S :=& \sum_{m=1}^{i} \binom{i}{k} (-c)^{i-m} T^m \\ =& (T-cI)^i - (-c)^i I; \end{aligned}$$

restricting S to N_i means $(T-cI)^i = 0$, and we have $S|_{N_i}$ in $B(N_i)$ is equal to $(-c)^i I$. The compactness of the identity operator means that $\dim N_i < \infty$.

□

References

[1] W. Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991.