Functional Analysis, Math 7321
Lecture Notes from March 30, 2017

taken by Wilfredo J. Molina

3.43 Theorem (Riesz-Fredholm (continuation))**.** *Let $T \in B(X)$ be compact, let $c \neq 0$, let $N_j = \ker (T - cI)^j$, and let $M_j = \operatorname{ran}(T - cI)$. Then*

a. $N_1 \subseteq N_2 \subseteq \cdots$ *and there is $k \in \mathbb{N}$ such that $N_k = N_{k+1} = N_{k+2} = \cdots$ and all N_j are invariant under T and of finite dimension.*

b. $M_1 \supseteq M_2 \supseteq \cdots$ *and there is $k' \in \mathbb{N}$ such that $M_{k'} = M_{k'+1} = M_{k'+2} = \cdots$ and all M_j are closed and of finite codimension.*

c. $k = k'$ *and M_k, N_k are complementary, $\left.(T - cI)\right|_{M_j}$ is invertible, and $\left.(T - cI)\right|_{N_j}$ is nilpotent.*

d. $\dim \ker (T - cI) = \operatorname{codim} \operatorname{ran}(T - cI)$.

Proof. □

b. Observe that $S = (T - cI)^i - (-c)^i I$ is compact since T is compact and that $M_i = \operatorname{ran}(T - cI)^i = \operatorname{ran}\left(s + (-c)^i I\right)$ is closed by the preceding lemma. Assume that $M_i \not\supseteq M_{i+1}$ for each i. Then we may pick $x_i \in M_i$ such that $\|x_i\| \leq 2$ and $\|x_i + M_{i+1}\| = 1$. If $x_j \in M_j$, then $Tx_j = (T - cI)\, x_j + cx_j \in Tx_j + M_j\, [\in (T - cI)\, x_j + M_j] \subseteq M_{j+1} + M_j \subseteq M_j$. So M_j is invariant under T.

If $i < j$, then

$$Tx_i - Tx_j = cx_i + (T - cI)\, x_i - Tx_j \in cx_i + M_{i+1} + M_j \in cx_i + M_{i+1} = c\,(x_i + M_{i+1})\,.$$

Moreover,

$$\|Tx_i - Tx_j\| \geq |c|\, \|x_i + M_{i+1}\| = |c| > 0.$$

So $(Tx_i)_{i \in \mathbb{N}}$ has no convergent subsequence, contradicting compactness.

c. Assume $a \in N_{k'+1}$, that is, $(T - cI)^{k'+1} a = 0$. Let $m > 0$ such that $m + k' \geq k$. From $(T - cI)^{k'} a \in M_{k'} = M_{k'+m}$, we get $(T - cI)^{k'} a = (T - cI)^{k'+m} b$ for some $b \in X$. Since $N_k = N_{k+1} = \cdots = N_{k'+m+1}$,

$$0 = (T - cI)^{k'+1} a = (T - cI)^{k'+m+1} b = (T - cI)^{k'+m} b = (T - cI)^{k'} a.$$

Therefore, $N_{k'+1} = N_{k'}$. By minimality of k, we get $k' \leq k$.

The same result holds for indices of T', denoted by $K_{T'}$ and $K'_{T'}$ such that $K'_{T'} \leq K_{T'}$. For the complementary inequality, note that

$$N_i^{\perp} = \left(\ker (T - cI)^i\right)^{\perp} = \overline{\operatorname{ran}(T' - cI)} = \operatorname{ran}(T' - cI)\,. \qquad (\star)$$

This follows from the closed range of compact perturbations of the identity. By minimality of $K'_{T'}$, $N_i \neq N_{i+1}$ for $i \leq K'_{T'}$. By comparing both sides, $K'_{T'}$ is the index for which $N^{\perp}_{K'_{T'}} = N^{\perp}_{K'_{T'+1}} = \cdots$, we deduce $N_k = N_{k+1} = N_{k+2}$ implies $K_T \leq K'_{T'}$ (We could also take the perps of both sides of $(\star)$. Then, whenever the right hand side stabilizes at $K'_{T'}$, the left hand side becomes N_i because it is a closed subspace.). Similarly, $\left(\ker (T - cI)^i\right)^{\perp} = \overline{\operatorname{ran}(T - cI)^i} = \operatorname{ran}(T - cI)^i$ implies $K_{T'} \leq K'_T$. Combining these inequalities yields $K_T \leq K'_{T'} \leq K_{T'} \leq K'_T$. Therefore, $K_T = K'_T$.

Let $x \in X'$. From $(T - cI)^k x \in M_k = M_{2k}$, there is $y \in X$ such that $(T - cI)^k x = (T - cI)^{2k} y$. Let $x = (T - cI)^k y + z$. Then $(T - cI)^k z = (T - cI)^k x - (T - cI)^{2k} y = 0$. So $z \in N_k$. Thus $X = M_k + N_k$. If $r \in M_k \cap N_k$, there is $s \in X$ such that $r = (T - cI)^k s$ and $0 = (T - cI)^k r = (T - cI)^{2r} s$. Therefore, $s \in N^{2k} = N_k$, and we conclude $r = (T - cI)^k s = 0$.

Let $x \in \ker (T - cI)|_{M_k}$. Then, by $x \in M_k$, there is $y \in X$ such that $x = (T - cI)^k y$ and from $(T - cI)^{k+1} y = 0$, we have $y \in N_{k+1} = N_k$. Therefore, $x = (T - cI)^k y = 0$, and $(T - cI)|_{M_k}$ is injective.

If $z \in M_k = M_{k+1}$, then $z = (T - cI)^{k+1} w = (T - cI)(T - cI)^k w$ for some w, which implies $z \in \operatorname{ran}(T - cI)|_{M_k}$. By the open mapping theorem, $(T - cI)|_{M_k}$ is invertible in $B(M_k)$. Finally, from $N_{k-1} \subseteq N_k$, there is $x \in N_k \setminus N_{k-1}$ such that $(T - cI)^{k-1}\Big|_{N_k} x \neq 0$. However, by the definition of N_k, $(T - cI)^k\Big|_{N_k} \equiv 0$. Therefore, $(T - cI)|_{N_k}$ is nilpotent with index k.

d. By above, $X = M_k \oplus N_k$. Moreover, $\infty > \dim\ker(T - cI) = \dim\ker(T - cI)|_{N_k} = \operatorname{codim}\operatorname{ran}(T - cI)|_{N_k} = \operatorname{codim}\operatorname{ran}(T - cI)$.