Functional Analysis, Math 7321
Lecture Notes from April 04, 2017

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Last time:We have completed direct sum decomposition with generalized eigen space.

2.44 Theorem. *Let X be separable Banach space and $T \in B(X)$ is compact, then*

(a) If $dim X = \infty$, $0 \in \sigma(T)$, if $c \in \sigma(T)\backslash\{0\}$, then c is an eigen value of finite multiplicity (i.e. eigen space has finite dimension).

(b) $\sigma(T)$ is an at most countable compact set with 0 being only possible accumulation point.

Proof. (a) From the ideal property of compact operators, if T is compact on an infinite-dimensional space X, then T is not invertible,because otherwise $I = T^{-1}T$ would be compact.

if $c \in \sigma(T)\backslash\{0\}$, $T - cI$ is not injective or not surjective, hence

$$\begin{aligned} 0 \le dim(ker(T - cI)) &= codim(ran(T - cI)) \\ &= dim(ker(T' - cI)) \text{ (by duality)} \\ &= codim(ran(T' - cI)) < \infty \end{aligned}$$

(Here we have used the result from the note on March 23 and 28, 2017)

(b) If $c \in \sigma(T)\backslash\{0\}$ then the generalized eigen space N_k as in Theorem on lecture note 03/30/2017 is closed and $S = (T - cI)\big|_{N_k}$ is nilpotent of index k in $B(N_k)$. For any $\alpha \neq 0$, from $S^k = 0$,

$$I = -\alpha^{-1}S^k + I = -\alpha^{-1}(S^k + \alpha^k I)$$

Assuming $(S - \alpha I)^{-1}$ is invertible, then

$$(S - \alpha I)^{-1} = -\alpha^{-1}(S^{k-1} + \alpha S^{k-2} + ... + \alpha^{k-1} I)$$

Moreover, we see that for $\alpha \neq 0$, multiplying the right hand side by $(S - \alpha I)$ shows that $(S - \alpha I)$ is invertible for $\alpha \neq 0$. So, for $z \neq c$, letting $\alpha = z - c \neq 0$, we get

$$(T - zI)\big|_{N_k} = S - \alpha I$$

is invertible.

Moreover, by the Theorem on lecture note 03/30/2017,

$$A = (T - zI)\big|_{M_k}$$

is invertible, so for $|\alpha| < \|A^{-1}\|^{-1}$, $A - \alpha I$ is invertible. And hence for $0 < |z - c| < \|A^{-1}\|^{-1}$, $(T - zI)$ is invertible on $X = N_k \oplus M_k$, and consequently , $z \notin \sigma(T)$.

Thus, c is an isolated point in $\sigma(T)$. By the positive distance between an element in $\sigma(T)\backslash\{0\}$ and all other elements, there is an open covers of disjoint balls of $\sigma(T)\backslash\{0\}$ and

$$vol(\cup_{\lambda\in\sigma(T)\backslash\{0\}} B_{\epsilon(\lambda)}(\lambda)) \subset vol(B_{2r(T)}(0))$$

So the number of such balls is at most countable.

□

We deduced the so-called Fredholm alternative.

2.45 Corollary. *Let X be a separable Banach space, $T \in B(X)$ is compact and $c \neq 0$ then either $(T - cI)$ is invertible or $0 < dim(ker(T - cI)) < \infty$.*

Proof. As from the above theorem (Theorem 11.1.2) we have

$$0 \leq dim(ker(T - cI)) = codim(ran(T - cI)) < \infty$$

if the $codim(ran(T-cI)) = 0$, then $(T-cI)$ is one-one and onto invertible by the open mapping theorem. Otherwise, $0 < dim(ker(T - cI)) < \infty$ from the above theorem (Theorem 11.1.2) because if $(T - cI)$ is not invertible then c is an eigen value of finite multiplicity, which means eigen space has finite dimension. □

In both of the cases, there is a non-trivial closed invariant subspace of T,

$$ker(T - cI) \;\; or \;\; ran(T - cI)$$

2.A Operators on Hilbert Spaces:

Given Hilbert Spaces H_1, H_2 and $T \in B(H_1, H_2)$, then the (Hilbert) adjoint $T^* \in B(H_1, H_2)$ satisfies for each $x \in H_1, y \in H_2$.

$$\langle Tx, y\rangle_{H_2} = \langle x, T^*y\rangle_{H_1}$$

This is the sesquilinear inner product.

Note:$(CT + S)^* = \overline{c}T^* + S^*$ because

$$\langle x, (cT + S)^*y\rangle = \overline{c}\langle x, T^*y\rangle + \langle x, S^*y\rangle.$$

2.46 Definition. Involution: An involution on a Banach Algebra B, is a map $i : B \mapsto B$ such that for $a, b \in B$, $c \in \mathbb{K}$

$$\begin{aligned} &i(i(a)) = a, \\ &i(a + b) = i(a) + i(b) \\ &i(ca) = \overline{c}\;\; i(a), \\ &(ab)^* = b^*a^* \end{aligned}$$

2.47 Definition. A C^*-Algebra is a Banach algebra with an involution i such that

$$\|i(x)x\| = \|x\|^2$$

2.48 Remarks. For $C*-$ Algebra, $\|i(x)\| = \|x\|$ because

$$\|x\|^2 = \|i(x)x\| \leq \|i(x)\|\|x\|\|$$

Thus, we have

$$\|x\| \leq \|i(x)\|$$

And,

$$\|i(x)\| \leq \|i(i(x))\| = \|x\|$$

Moreover, if $x_n \to x$ then

$$\|x_n - x\| = \|i(x_n) - i(x)\| \to 0.$$

So, i is continuous.

2.49 Theorem. *Let H_1, H_2, be Hilbert spaces, then $T \in B(H_1, H_2)$ satisfies:*

$$\|T^*T\| = \|T\|^2.$$

Proof. We recall

$$\begin{aligned}\|T\| &= \sup_{\|x\|\leq 1} \|Tx\| \\ &= \sup_{\|x\|,\|y\|\leq 1} |\langle Tx, y\rangle| \\ &= \sup_{\|x\|,\|y\|\leq 1} |\langle x, T^*y\rangle| \\ &= \sup_{\|x\|,\|y\|\leq 1} |\langle T^*y, x\rangle| \\ &= \|T^*\|\end{aligned}$$

So,

$$\|T^*T\| \leq \|T^*\|\|T\|$$

Conversely, for x, with $\|x\| \leq 1$

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle T^*Tx, x\rangle \\ &\leq \|T^*Tx\|\|x\| \text{ (By Cauchy Schwarz inequality)} \\ &\leq \|T^*T\|\|x\|\|x\| \\ &= \|T^*T\|\|x\|^2\end{aligned}$$

Thus, we have now

$$\|T\|^2 \leq \|T^*T\|$$

From above two case, we conclude that

$$\|T^*T\| = \|T\|^2.$$

□

Polarization identity: Let $\|x\|$ denotes the norm of vector x and $\langle x,y\rangle$ be the inner product of vectors x,y and let V be a vector space then the polarization identity to define $\langle x,y\rangle$:

If $\mathbf{V}=\mathbb{R}$,

$$\langle x,y\rangle=\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right)\forall x,y\in V.$$

If $\mathbf{V}=\mathbb{C}$,

$$\langle x,y\rangle=\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2+\imath\|x+\imath y\|^2-\imath\|x-\imath y\|^2\right),\forall x,y\in V.$$

2.50 Lemma. *Given $T\in B(H)$, H be a complex Hilbert space then $T=0$ iff $\langle Tx,x\rangle=0$.*

Proof. Let T is characterized by values $\langle Tx,y\rangle$ for $x,y\in H$. Then by using the polarization identity we have $\forall x,y\in H$,

$$\langle Tx,y\rangle=\frac{1}{4}[\langle T(x+y),(x+y)\rangle-\langle T(x-y),(x-y)\rangle+i\langle T(x+iy),(x+iy)\rangle-i\langle T(x-iy),(x-iy)\rangle)].$$

Since T is linear then we have,

$$\begin{aligned}\langle Tx,y\rangle=&\frac{1}{4}[\langle Tx,x\rangle+\langle Ty,x\rangle+\langle Tx,y\rangle+\langle Ty,y\rangle-\langle Tx,x\rangle+\langle Ty,x\rangle+\langle Tx,y\rangle-\langle Ty,y\rangle+\\&i(\langle Tx,x\rangle+i\langle Ty,x\rangle-i\langle Tx,y\rangle+\langle Ty,y\rangle)-i(\langle Tx,x\rangle+i\langle Ty,x\rangle-i\langle Tx,y\rangle+\langle Ty,y\rangle)]\end{aligned}$$

Which means,

$$\begin{aligned}\langle Tx,y\rangle=&\frac{1}{4}[\langle Tx,x\rangle+\langle Ty,x\rangle+\langle Tx,y\rangle+\langle Ty,y\rangle-\langle Tx,x\rangle+\langle Ty,x\rangle+\langle Tx,y\rangle-\langle Ty,y\rangle+\\&i\langle Tx,x\rangle-\langle Ty,x\rangle+\langle Tx,y\rangle+i\langle Ty,y\rangle)-i\langle Tx,x\rangle+\langle Ty,x\rangle-\langle Tx,y\rangle-i\langle Ty,y\rangle]\end{aligned}$$

Thus, we get

$$\langle Tx,y\rangle=\frac{1}{4}[2\langle Tx,y\rangle+2\langle Ty,x\rangle]=\frac{1}{2}[\langle Tx,y\rangle+\langle Ty,x\rangle]$$

This implies,

$$\langle Tx,y\rangle=0\ \text{ for all } x,y\in H$$

Substituting $y=Tx$ we get,

$$\langle Tx,Tx\rangle=0\ \text{ for all } x\in H.\Rightarrow\|Tx\|^2=0\ \text{ for all } x\in H.$$

Thus, $T=0$.

Conversely, to show that $\langle Tx,x\rangle=0$ when $T=0$, we have from the polarization identity

$$\langle Tx,y\rangle=\frac{1}{4}[\langle T(x+y),(x+y)\rangle-\langle T(x-y),(x-y)\rangle+i\langle T(x+iy),(x+iy)\rangle-i\langle T(x-iy),(x-iy)\rangle)].$$

Substituting $y=x$ above we get and by linearity T we get

$$\begin{aligned}\langle Tx,x\rangle=&\frac{1}{4}[\langle T(x+x),(x+x)\rangle-\langle T(x-x),(x-x)\rangle+\\&i\langle T(x+ix),(x+ix)\rangle-i\langle T(x-ix),(x-ix)\rangle)]\\=&\frac{1}{4}[\langle Tx,x\rangle+\langle Tx,x\rangle+\langle Tx,x\rangle+\langle Tx,x\rangle-\langle Tx,x\rangle+\langle Tx,x\rangle+\langle Tx,x\rangle-\langle Tx,x\rangle+\\&i\langle Tx,x\rangle-\langle Tx,x\rangle+\langle Tx,x\rangle+i\langle Tx,x\rangle)-i\langle Tx,x\rangle+\langle Ty,x\rangle-\langle Tx,x\rangle-i\langle Tx,x\rangle]\end{aligned}$$

Since $T = 0$ implies

$$\langle Tx, x \rangle = 0$$

$\square$

2.B Orthogonal Projection:

2.51 Definition. A projection $P = P^2$ in $B(H)$ is orthogonal if $ker(P) \perp ran(P)$.

2.52 Theorem. *If P is a non-zero projection, then the following are equivalent*

(a) P *is orthogonal*

(b) $P = P^*$

(c) $\|P\| = 1$

Proof. First, $(a) \implies (b)$, If P is orthogonal then, $ran(P) \perp ran(I - P)$. Thus, for each $x \in H$,

$$\begin{aligned} &\langle Px, (I - P)x \rangle = 0. \\ \implies &\langle (I - P)^* Px, x \rangle = 0. \\ \implies &(I - P)^* P = 0. \\ \implies &P = P^* P \end{aligned}$$

Thus,

$$P^* = (P^* P)^* = P^* P^{**} = P^* P = P \text{ from above relation}$$

Now, to show $(b) \implies (c)$ since we have $P = P^*$ which gives

$$\begin{aligned} \|Px\|^2 &= \langle Px, Px \rangle \\ &= \langle P^* Px, x \rangle \\ &= \langle P^2 x, x \rangle \text{ because } P^* = P \\ &= \langle Px, x \rangle \text{ because } P^2 = P \text{ by defination} \\ &\leq \|Px\| \|x\| \text{ by Cauchy Schwarz Inequality} \end{aligned}$$

Thus, $\|P\| \leq 1$ by taking $\max \|X\| \leq 1$.

Choosing $x \in ran(P) \backslash \{0\}$ gives $Px = x$ then combining both the results we get,

$$\|P\| = 1.$$

Finally, to show $(c) \implies (a)$ we show this by contrapositive, for this assume P is not orthogonal. Then there are $x, y \in H$ with $\|x\| = \|y\| = 1$, and $x \in ran(P), y \in ker(P)$,

$$\langle x, y \rangle \neq 0$$

WLoG, assume $\langle x, y \rangle = -t < 0$ otherwise replace x by λx, $|\lambda| = 1$.

Let $z = x + ty$ then

$$\begin{aligned}\|z\|^2 &= \|x\|^2 + 2t\langle x, y\rangle + t^2\|y\|^2 \\ &= 1 - t^2 \text{ because } \|x\| = 1 = \|y\| \quad \text{and} \quad \langle x, y\rangle = -t \\ &< 1 = \|x\|^2 = \|Pz\|^2\end{aligned}$$

This shows that

$$\|z\|^2 < \|Pz\|^2$$

This implies

$$\|P\| > 1$$

Hence by contrapositive we got that if $\|P\| = 1 \implies P$ is orthogonal. □