Lecture Notes from January 17, 2023

taken by Bernhard Bodmann

0 Course Information

Class:	Tu&Th 10am-11:20pm, SW 219
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Office:	PGH 604; Tu 1-2pm, We 2-3pm
Content:	This course is the second part of a two semester sequence covering main advanced results in functional analysis, including Hilbert spaces, Banach spaces, and linear operators on theses spaces. Functional analysis combines two fundamental branches of mathematics: analysis and linear algebra. Limiting arguments from analysis become essential in order to resolve questions from linear algebra in infinite-dimensional spaces. In addition, there are close connections between algebraic and topological properties in such spaces that deepen our understanding even in the finite dimensional case. Topics covered in this second part of the course sequence include: Spectral theory in Banach algebras, C^* algebras, properties of the spectrum, Gelfand's representation theory, properties of commutative C^*-algebras, functional calculus, positivity, states, spectral theory for bounded normal operators.
Prerequisites:	Apart from the official prerequisites, you should have seen the Hahn Banach Theorem, the Uniform Boundedness Theorem together with its consequence for operator sequences, the Banach-Steinhaus Theorem, and the Open Mapping Theorem, all in the context of normed spaces. For details, see the course handout from August 23, 2022.
Text:	Walter Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991.
Assignments:	You will be asked take notes and typeset them in LaTeX.
Final Grade:	Based on the quality of notes. The goal of the notes is to absorb the material presented in class and prepare the notes so that one of your (hypothetical) peers who missed class will be able to follow your explanations and learn what happened in class.

All of the course-related information is listed in the official syllabus, which can be found on the website for our course:

www.math.uh.edu/~bgb/Courses

1 Summary of spectral theory via the representation of involutive semigroups

We begin with a review of the material from the last semester.

We recall an example of an involutive semigroup.

1.1 Example. Let $S = (\mathbb{N}_0 \times \mathbb{N}_0, +)$ with $(n, m)^* = (m, n)$. A representation π of this involutive semigroup is given by selecting $A \in B(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space, such that A is normal, so $AA^* = A^*A$, and then letting

$$\pi(n, m) = A^n (A^*)^m .$$

1.2 Definition. A representation π of an involutive semigroup S is called non-degenerate if

$$\overline{\pi(S)\mathcal{H}} = \mathcal{H} .$$

We recall that we can reduce the representation of an involutive semigroup to a closed subspace of $\mathcal{H}$ without losing any information, making it non-degenerate.

1.3 Theorem. *Let $\mathcal{H}_0 = \{v \in \mathcal{H} : (\forall s \in S)\pi(s)v = 0\}$ then $\mathcal{H}_0$ is closed and invariant under S, and so is the orthogonal complement $\mathcal{H}_0^\perp$, and $\pi|_{\mathcal{H}_0^\perp}$ is non-degenerate.*

Having extracted the essential part of a representation by reducing to a non-degenerate component, we can ask if it can be reduced further.

1.4 Theorem. *The representation of an involutive semigroup S is non-degenerate if and only if it is the direct sum of cyclic representations.*

In the finite-dimensional abelian case, this simplifies even more.

1.5 Theorem. *If S is an abelian involutive semigroup, then each finite-dimensional irreducible representation is one-dimensional.*

Next, we connect between representation theory and spectral theory.

1.6 Definition. The representations (homomorphisms) $\pi : S \to B(\mathbb{C}) \simeq \mathbb{C}$ of an involutive semigroup form a set $\widehat{S}_0$ and we call $\widehat{S} = \widehat{S}_0 \setminus \{0\}$ the characters of S.

The first main result on spectral theory is the decomposition of the Hilbert space in eigenspaces of a normal operator.

1.7 Theorem. *Let* A *be a normal operator on a finite-dimensional Hilbert space* $\mathcal{H}$*, and* $\pi(n,m) = A^n(A^*)^m$ *as before, then*

$$\mathcal{H} = \bigoplus_{\chi \in \widehat{S}_0} \mathcal{H}_\chi$$

with

$$\mathcal{H}_\chi = \{v \in \mathcal{H} : (\forall s \in S)\pi(s)v = \chi(s)v\},$$

and $\widehat{S}_0 = \mathbb{C}$*,* $\chi(n,m) = \lambda^n \overline{\lambda}^m$ *for* $\lambda \in \mathbb{C}$*.*

We generalize from the spectrum of an operator to the spectrum of an entire Banach algebra.

1.8 Definition. Let $\mathcal{A}$ be a complex Banach algebra and if $\mathcal{A}$ does not have a unit, then $\widetilde{\mathcal{A}}$ the algebra with the unit, then we call for $a \in \mathcal{A}$

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } \widetilde{\mathcal{A}}\}$$

the spectrum of the element a.

We let $\Gamma_\mathcal{A}$ be the space of all (continuous) non-trivial homomorphisms to $\mathbb{C}$,

$$\Gamma_\mathcal{A} = \{\chi : \mathcal{A} \to \mathbb{C}\}$$

and call this set the spectrum of $\mathcal{A}$.

The reason for calling $\Gamma_\mathcal{A}$ the spectrum becomes apparent in the following theorem.

1.9 Theorem. *Let* $\mathcal{A}$ *be a commutative Banach algebra with unit, then* $\Gamma_\mathcal{A}$ *is non-empty and*

$$\mathcal{G} : \mathcal{A} \to C(\Gamma_\mathcal{A}), x \mapsto \hat{x}$$

with $\hat{x}(\chi) \equiv \chi(x)$ *is a homomorphism with* $\hat{x}(\Gamma_\mathcal{A}) = \sigma(x)$ *and* $\|\hat{x}\|_\infty = r(x)$*, where* $r(x)$ *is the spectral radius of* x*. Furthermore,* $\hat{1}$ *is the unit in* $C(\Gamma_\mathcal{A})$*.*