Lecture Notes from January 19, 2023

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Groups, connectedness and Banach algebras

We consider the group of a Banach algebra and connect between topological and group theoretic properties.

1.2 Proposition. *Let $\mathcal{A}$ be a Banach algebra and G_0 the connected component of the identity of $G(\mathcal{A})$, then G_0 is a normal subgroup of $G(\mathcal{A})$.*

Proof. We recall that G_0 is the smallest set in $G(\mathcal{A})$ that is open and closed and contains the identity. By continuity of the group multiplication, if $b \in G(\mathcal{A})$, then the coset bG_0 is connected. Assuming $b, c \in G_0$, then bG_0 is connected and contains b and bc. Thus, $G_o \cup bG_0$ is connected in $G(\mathcal{A})$, hence $G_0 \cup bG_0 \subset G_0$ and we see G_0 is closed under multiplication. Similarly, if $b \in G_0$, then $b^{-1}G_0 \cup G_0 \subset G_0$, and we deduce G_0 is a subgroup of $G(\mathcal{A})$. Again by continuity of multiplication, for any $b \in G(\mathcal{A})$, $b^{-1}G_0 b$ is connected, open and closed and contains the identity, so $b^{-1}G_0 b = G_0$. □

Because of G_0 being a normal subgroup, we can build $G(\mathcal{A})/G_0$ and investigate the structure of this quotient group. At first, we investigate the structure of G_0 more closely.

1.3 Definition. We use power series to define for $b \in B_1(0) \subset \mathcal{A}$

$$\ln(1+b) = -\sum_{n>0} \frac{1}{n}(-b)^n$$

and for $a \in \mathcal{A}$

$$e^a = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \, .$$

1.4 Lemma. *If $\|1-b\| < 1$, then*

$$e^{\ln b} = b$$

so $B_1(1) \subset e^{\mathcal{A}}$.

Proof. Summing doubly indexed power series. □

We deduce a result on constructing G_0.

1.5 Theorem. *Let $\mathcal{A}$ be a Banach algebra with unit, then G_0 consists of $\mathcal{F}$, finite products of elements from $e^{\mathcal{A}}$.*

Proof. Using the power series summation, we see $e^a e^{-a} = e^{a-a} = 1$, so $e^{\mathcal{A}} \subset G(\mathcal{A})$. The power series also shows e^{ta} interpolates continuously between $e^{0a} = 1$ and $e^{1a} = e^a$, so $e^{\mathcal{A}}$ is (even arc-wise) connected. This implies $e^{\mathcal{A}} \subset G_0$. $\mathcal{F}$ is a subgroup of G_0. By $B_1(1) \subset \mathcal{F}$, it is open. Cosets of $\mathcal{F}$ are open as well, so $\mathcal{F}$ is closed, thus $\mathcal{F} = G_0$. □

1.6 Corollary. *If $\mathcal{A}$ is a commutative Banach algebra with unite, then $e^{\mathcal{A}} = G_0$.*

After studying the structure of G_0, we can investigate $G(\mathcal{A})/G_0$.

1.7 Definition. The group $\Lambda_{\mathcal{A}} = G(\mathcal{A})/G_0$ is called the index group of $\mathcal{A}$.

We consider a particular example $\mathcal{A} = C(X)$ for a compact Hausdorff space X.

1.8 Example. Let X be a compact Hausdorff space and $\mathcal{A} = C(X)$, then $G(\mathcal{A}) = \{f \in C(X) : X \mapsto \mathbb{C} \setminus \{0\}\}$. From our above corollary, if $f \in G_0$, then there is $g \in C(X)$ such that $f = e^g$. Now we can define for $t \in [0, 1]$ the function family given by $f_t(x) = e^{tg(x)}$, which interpolates continuously between $f_0 = 1$ and $f_1 = f$. Thus, f is homotopic to 1. Conversely, if f is homotopic to one, then $f \in G_0$. Similarly, $f_1 \sim f_2$ (belong to the same coset of G_0) if and only if f_1 and f_2 are homotopic.

We conclude the index group of $C(X)$ is concretely given by the set of homotopy classes of $C(X, \mathbb{C} \setminus \{0\})$, with the group operation being pointwise multiplication.